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AdS/CFT and Integrability: a Review

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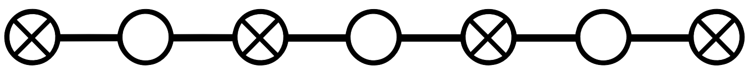

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1 Introduction

One of the biggest breakthroughs in theoretical physics in the past few decades is undoubtedly the discovery of the AdS/CFT correspondence [1], which relates a specific gauge field theory, $\mathcal{N} = 4$ super Yang-Mills, to a type IIB string theory on an $AdS_5 \times S^5$ background [2]. Though both of these theories are in some sense special and at first sight don't have anything to do with the real world, the correspondence has received a tremendous amount of attention and is being actively researched upon even to this day. There are many important reasons for that. First of all, the correspondence relates a theory with gravity (string theory) to a theory without, which may eventually shed light on the nature of gravity. It is also a realization of another idea in physics, the holographic principle, which states that in certain cases all information about a physical system can be embedded into a space of lesser dimensionality [3]. But what is probably the most attractive feature of the correspondence is that it is a strong/weak duality, i.e. it translates problems at strong coupling in a gauge theory to problems of weakly interacting strings. This is important, because it provides a possible tool for solving real life problems in the theory of strong interactions, QCD, where at low energies we have a very strongly coupled theory and perturbation theory does not work anymore.

The idea that QCD is related to strings is not new, in fact, string theory started out as a candidate theory for the strong interactions by trying to explain the so-called Regge trajectories observed in experiments [4]. String theory fell out of favor once it was discovered that QCD correctly describes the strong interactions only to reemerge later with bigger ambitions – to unify gravity with all known fundamental interactions and produce a theory of everything. Yet as it turned out that doing calculations in QCD is notoriously difficult in the low energy regime, attention shifted back to string theory in hopes of coming up with new ideas. Indeed, the strong interactions show a lot of string like behavior, e.g. asymptotic freedom can be intuitively explained by a string holding quarks together – the closer they are, the less they interact and vice versa. It was 't Hooft who first noticed that the planar limit of a gauge theory, i.e. taking N, the number of colors, to infinity produces something like a string theory – the planar Feynman diagrams can be interpreted as two dimensional world sheets of interacting strings [5]. But it was the work of Maldacena in 1998 that really sparked the new revolution that is now called AdS/CFT duality [2]. Even though the duality can be extended beyond the original example (e.g. in [6]), it has remained the most actively explored one – due to its simplicity it makes a perfect playground for exploring new ideas.

Theories on both sides of the duality are very symmetric: $\mathcal{N} = 4$ SYM is a conformal field theory with 16 super charges and an SU(N) gauge symmetry and type IIB string theory on $AdS_5 \times S^5$ can be formulated as a coset space non-linear sigma model [7]. In fact there is so much symmetry, that it turns out to be possible to solve these theories exactly in the planar limit [8]. This phenomenon goes under the name of *integrability* and has been under active research for the last decade. What it means to solve a theory exactly is a topic on its own and it will be addressed in this thesis. In short, it all started by noticing that at one loop level the dilatation operator of $\mathcal{N} = 4$ SYM can be identified with the Hamiltonian of a one dimensional ferromagnetic spin chain [9]. Such spin chains are known to be integrable, meaning that one can find the energy levels of all the states in the spin chain exactly. For gauge theory this means that one can find anomalous dimensions for all field states in the theory. The remarkable thing is that it is possible to identify the dilatation operator with the Hamiltonian of some integrable spin chain at *all loop level*. This means that it is possible to find the exact anomalous dimensions of all operators in the gauge theory at any coupling. If this were possible in QCD, we would be able to calculate hadron masses from first principles and they would be given in some closed algebraic form. This would be a dream come true for theoretical physics.

Integrability is wonderful for another reason, namely that it allows to check the AdS/CFT correspondence, which is formally still a conjecture. If the correspondence is correct, the string theory dual of $\mathcal{N} = 4$ SYM should also be integrable. What this means is that if it is possible to find anomalous dimensions of operators in gauge theory, it should be possible to do the same with energy levels of strings. And indeed this turns out to be the case [10]. The energies of string solutions can be found in some closed algebraic form in terms of the string coupling. Again, this is a non-perturbative result, which means that one can compare predictions from string theory to predictions from gauge theory directly, i.e. at the same value of the coupling. Since AdS/CFT is a strong/weak duality, it is also possible to compare e.g. perturbation theory results from string theory to exact results from gauge theory. An enormous amount of computational checks have been carried out in such a manner and the results have been phenomenal – all predictions agree with unprecedented accuracy, confirming that the AdS/CFT duality must be correct at least in the planar limit.

This thesis is a review of AdS/CFT and integrability, it should be noted that it is by no means original. It consists of two major parts. In the first part we present a pedagogical introduction to the AdS/CFT correspondence, focusing attention to the theories on both sides of the correspondence by exploring their contents and symmetries. In the second part we introduce integrable structures present in both theories and discuss their significance in the context of the AdS/CFT correspondence. We use the already mentioned spectral problem as the canonical example for applying integrability to AdS/CFT. More or less following the historic path we show how the spectral problem can be solved in various limits by Bethe ansatz techniques culminating in the full all-loop solution from the gauge theory side. We then switch to string theory and approach the same problem from there. The spectral curve method is discussed, which is basically the Bethe ansatz analogue in string theory. We conclude by showing how solutions to the spectral problem in both theories are related to one another, hinting that they actually are the same solution simply approached from different limits of the theory. We finish the thesis with a section on further developments and open problems in the rapidly advancing field of integrability. These include solving AdS/CFT completely and generalizing the discovered techniques to more realistic theories like QCD.

2 The AdS/CFT correspondence

In this section we review the original Maldacena AdS/CFT correspondence. Though there are other gauge/gravity dualities discovered, the first one still remains the most popular one mainly due to its simplicity. Before reviewing the correspondence we start by introducing theories on both sides of the correspondence, i.e. $\mathcal{N} = 4$ SYM and type IIB string theory on an $AdS_5 \times S^5$ background and its low energy supergravity limit. We end the section by discussing the correspondence itself without going into much detail, instead we focus on the integrable structures found in theories on both sides of the correspondence and the implications of integrability to AdS/CFT in the next section.

2.1 $\mathcal{N} = 4$ Super Yang-Mills theory

$\mathcal{N} = 4$ Super Yang-Mills theory is a quantum field theory much like the Standard Model of particle physics with a certain field content and interaction pattern. What is special about it is the amount of symmetry available – not only is it a supersymmetric gauge theory, it is also conformally invariant both at the classical and quantum levels, i.e. it is a conformal field theory. In four dimensions a field theory with 16 supercharges is uniquely determined by specifying the gauge group, the fields then live in the vector multiplet of the supersymmetry algebra and in the adjoint of the gauge group. The action is given by [11]

$$\begin{aligned} S = \int d^4x \ \mathrm{Tr} \Big(&-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \mathcal{D}_\mu \Phi_I \mathcal{D}^\mu \Phi^I - \bar{\psi}^a \sigma^\mu \mathcal{D}_\mu \psi_a \\ &- \frac{ig}{2} \sigma_I^{ab} \psi_a [\Phi^I, \psi_b] - \frac{ig}{2} \sigma^I_{ab} \bar{\psi}^a [\Phi_I, \bar{\psi}^b] - \frac{g^2}{4} [\Phi_I, \Phi_J][\Phi^I, \Phi^J] \Big) \end{aligned} \tag{2.1}$$

where $\mu = 1, ..., 4$ as usual, $I, J = 1, ..., 6$, $a, b = 1, ..., 4$ and σ_μ and σ_I are the chiral versions of the gamma matrices in four and six dimensions respectively. The covariant derivative $\mathcal{D}_\mu$ is defined as

$$\mathcal{D}_\mu = \partial_\mu - ig \ [A_\mu, \ ]. \tag{2.2}$$

Alternatively the action can be formulated as a $\mathcal{N} = 1$ supersymmetric gauge theory in 10 dimensions with the action given by

$$S = \int d^{10}x \, \mathrm{Tr} \left(-\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} \Psi \Gamma^M \mathcal{D}_M \Psi \right) \tag{2.3}$$

where Ψ is now a Majorana-Weyl spinor in 16 dimensions and Γ^M is the 16 dimensional gamma matrix. The action (2.1) can be recovered by dimensional reduction to four dimensions. The gauge field A_M decomposes to the four dimensional gauge field A_μ and to six real scalar fields Φ_I whereas the Majorana-Weyl spinor Ψ_A breaks up into four copies of the left and right Weyl spinors in four dimensions

$$\Psi_A \; (A = 1, ..., 16) \;\; \rightarrow \;\; \bar{\psi}^a_{\dot{\alpha}}, \; \psi_{a\alpha} \; (\alpha, \dot{\alpha} = 1, 2, \;\; a = 1, ..., 4). \tag{2.4}$$

This theory has an additional SU(4) $\simeq$ SO(6) symmetry called *R-symmetry* that permutes the scalars, which live in the fundamental representation of SO(6) and the spinors, which live in the fundamental of SU(4). From this it follows that we can combine the six real scalars Φ^I into three complex scalars ϕ^{ab}, which then transform under the second rank antisymmetric representation of SU(4). The gauge field is a singlet under R-symmetry.

It is now a straightforward but rather tedious task to calculate the beta function for this theory. For any SU(N) gauge theory at one loop level it is given by [12]

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N - \frac{1}{6}\sum_s C_s - \frac{1}{3}\sum_f \tilde{C}_f\right) \tag{2.5}$$

where the first sum is over the real scalars and the second one over the fermions. C_s and $\tilde{C}_f$ are the quadratic casimirs, which in our case are equal to N since all fields are in the adjoint representation of the group. It is then easy to see that at least at one loop level the theory is conformally invariant. It can be shown that the β function is identically zero to all orders in perturbation theory [13], hence the theory is fully conformally invariant even after quantization. After discussing the full symmetry algebra of the theory and its representations we will give an elegant argument why this is true.

2.1.1 Symmetry group of the theory

Conformal symmetry, supersymmetry and R-symmetry are a part of a bigger group PSU(2, 2|4), which is known as the $\mathcal{N} = 4$ *superconformal group*. It is the full symmetry group of $\mathcal{N} = 4$ SYM and is unbroken by quantum corrections [14]. Hence studying it and its representations further can provide more insights into the theory itself. It is an example of a *supergroup*, i.e.

a graded group containing bosonic and fermionic generators. The theory of supergroups is highly developed (see [15]) and much of the techniques from studying simple groups carry over to supergroups with some additional complications, i.e. Dynkin diagrams, root spaces, weights etc. The cover sheet of this thesis features the Dynkin diagram of PSU$(2,2|4)$.

PSU$(2,2|4)$ has the bosonic subgroup of SU$(2,2)\times$SU(4), where SU$(2,2)\simeq$ SO$(2,4)$ is the conformal group in four dimensions and SU$(4)\simeq$ SO(6) is the R-symmetry. The conformal group has the Poincaré group as a subgroup, which has 10 generators P_μ and $M_{\mu\nu}$, in addition there is the generator for dilatations D and four special conformal generators K_μ. Their commutation relations are given by [14]

$$\begin{gathered}[D,M_{\mu\nu}]=0 \quad [D,P_\mu]=-iP_\mu \quad [D,K_\mu]=+iK_\mu \\ [M_{\mu\nu},P_\lambda]=-i(\eta_{\mu\nu}P_\nu-\eta_{\lambda\nu}P_\mu) \quad [M_{\mu\nu},K_\lambda]=-i(\eta_{\mu\lambda}K_\nu-\eta_{\lambda\nu}K_\mu) \\ [P_\mu,K_\nu]=2i(M_{\mu\nu}-\eta_{\mu\nu}D). \end{gathered} \tag{2.6}$$

$\mathcal{N}=4$ supersymmetry has 16 supercharges $Q_{a\alpha}$ and $\tilde{Q}^a_{\dot\alpha}$ where $\alpha,\dot\alpha=1,2$ are the Weyl spinor indices and $a=1,...,4$ are the R-symmetry indices. These generators have the usual commutation and anti-commutation relations with the Poincaré generators given by

$$\begin{gathered}\{Q_{\alpha a},\tilde{Q}^b_{\dot\alpha}\}=\gamma^\mu_{\alpha\dot\alpha}\delta^b_a P_\mu \quad \{Q_{\alpha a},Q_{\alpha b}\}=\{\tilde{Q}^a_{\dot\alpha},\tilde{Q}^b_{\dot\alpha}\}=0 \\ [M^{\mu\nu},Q_{\alpha a}]=i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma a} \quad [M^{\mu\nu},\tilde{Q}^a_{\dot\alpha}]=i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\tilde{Q}^a_{\dot\gamma} \\ [P_\mu,Q_{\alpha a}]=[P_\mu,\tilde{Q}^b_{\dot\alpha}]=0 \end{gathered} \tag{2.7}$$

where $\gamma^{\mu\nu}_{\alpha\beta}=\gamma^{[\mu}_{\alpha\dot\alpha}\gamma^{\nu]}_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$. Commutators between supercharges and the conformal generators are also non trivial and even introduce new supercharges,

$$\begin{gathered}[D,Q_{\alpha a}]=-\tfrac{i}{2}Q_{\alpha a} \quad [D,\tilde{Q}^a_{\dot\alpha}]=-\tfrac{i}{2}\tilde{Q}^a_{\dot\alpha} \\ [K^\mu,Q_{\alpha a}]=\gamma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\tilde{S}_{\dot\beta a} \quad [K^\mu,\tilde{Q}^a_{\dot\alpha}]=\gamma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}S^a_\beta \end{gathered} \tag{2.8}$$

where $\tilde{S}_{\dot\alpha a}$ and S^a_α are the *special conformal supercharges*. They have a reversed Weyl/R-symmetry representation matching from the usual supercharges and together with them bring the total of supercharges to 32. The commutation and anti-commutation relations for the special conformal supercharges are very much like the ones for normal supercharges,

$$\begin{gathered}\{S^a_\alpha,\tilde{S}_{\dot\alpha b}\}=\gamma^\mu_{\alpha\dot\alpha}\delta^a_b K_\mu \quad \{S^a_\alpha,S^b_\alpha\}=\{\tilde{S}_{\dot\alpha a},\tilde{S}_{\dot\alpha b}\}=0 \\ [M^{\mu\nu},S^a_\alpha]=i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}S^a_\gamma \quad [M^{\mu\nu},\tilde{S}_{\dot\alpha a}]=i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\tilde{S}_{\dot\gamma a} \\ [K_\mu,S^a_\alpha]=[K_\mu,\tilde{S}_{\dot\alpha a}]=0. \end{gathered} \tag{2.9}$$

Finally the anti-commutation relations between the special conformal and usual supercharges close the algebra,

$$\begin{aligned}
\{Q_{\alpha a}, S^b_\beta\} &= -i\epsilon_{\alpha\beta}\sigma^{IJ}{}_a{}^b R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta_a{}^b M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta_a{}^b D \\
\{\tilde{Q}^a_{\dot{\alpha}}, \tilde{S}_{\dot{\beta}b}\} &= +i\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{IJa}{}_b R_{IJ} + \gamma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\delta^a{}_b M_{\mu\nu} - \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\delta^a{}_b D \\
\{Q_{\alpha a}, \tilde{S}_{\dot{\beta}b}\} &= \{\tilde{Q}^a_{\dot{\alpha}}, S^b_\beta\} = 0
\end{aligned} \tag{2.10}$$

where R_{IJ} are the generators of R-symmetry with $I, J = 1, ..., 6$. All supercharges transform under the two spinor representations of the R-symmetry group and all other generators commute with it.

Looking at the commutation relations of the conformal subgroup (2.6), we see that the operators P_μ and K_μ act as raising and lowering operators for the dilatation operator D – this gives a hint as to how we could construct representations of the group. The dilatation operator D is the generator of scalings, i.e. upon a rescaling $x \to \lambda x$ a local operator in a field theory scales as

$$\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x) = \lambda^{-iD}\mathcal{O}(x)\lambda^{iD} \tag{2.11}$$

where Δ is the *conformal dimension* of the operator $\mathcal{O}(x)$. Classically it is simply the energy dimension of the operator, but as we will see later it can (and often does) get quantum corrections. Restricting to the point $x = 0$, which is a fixed point of scalings, we see that the conformal dimension is the eigenvalue of the dilatation operator,

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0). \tag{2.12}$$

It is now clear that acting on a field with K_μ should lower the dimension by one and acting with P_μ – raise it by one. We can show this explicitly using the Jacobi identity. Since operators in a unitary quantum field theory should have positive dimensions (aside from the identity operator), we should not be able to keep lowering the dimension indefinitely, i.e. there should always be an operator that satisfies

$$[K_\mu, \tilde{\mathcal{O}}(0)] = 0. \tag{2.13}$$

We call such operators *primary operators*. Acting on these with P_μ keeps producing operators with a dimension one higher – we call these the *descendants* of $\tilde{\mathcal{O}}(0)$. We can also act with the supercharges and looking at the commutators in (2.8) we see that they raise the dimension by 1/2, while the special conformal supercharges lower it by 1/2. A primary operator along with its descendants makes up an irreducible representation of PSU$(2,2|4)$, which is infinite dimensional. However we can get smaller representations by requiring that operators commute with some of the supercharges, i.e.

$$[Q_{\alpha a}, \tilde{\mathcal{O}}(0)] = 0 \tag{2.14}$$

for some α, a. Using the algebra one can show that there is a class of operators that satisfy the condition $J = \Delta$, where J is the charge for any of the 3 R-symmetry generators in the Cartan algebra of SO(6). It can also be shown that they commute with half of the supercharges. Such operators are called *chiral primary* or half BPS operators.

As stated before, an operator's conformal dimension may get renormalized through quantum corrections, i.e. it acquires an *anomalous dimension*, which depends on the gauge coupling of the theory. An important fact is that operators in the same PSU$(2,2|4)$ representation must have the same anomalous dimension, because the generators of the group can only change it by half integer steps and there's only a finite number of generators. What is more, chiral primary operators are protected from quantum corrections, because at any coupling the total dimension is related to some charge of SO(6) by $\Delta = J$, but charges of compact groups are quantized, meaning that the dimension can't continuously depend on the coupling and hence the anomalous dimension must vanish. As an example of a chiral primary operator we can take the trace of L complex scalar fields, e.g. $Z = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$,

$$\Psi = \mathrm{Tr}[Z^L], \quad L \geq 2. \tag{2.15}$$

This is a chiral primary, since the classical dimension of Ψ is L and it has the R-charges $[L, 0, 0]$, hence it satisfies the chiral primary condition. We will have more to say about these single trace local operators when we discuss integrability. In a similar fashion it can be shown that the operators $\mathrm{Tr}\,\mathcal{F}_+\mathcal{F}_+$ and $\mathrm{Tr}\,\mathcal{F}_-\mathcal{F}_-$, where $\mathcal{F}_+$ and $\mathcal{F}_-$ are the self-dual and anti self-dual field strengths, are also chiral primaries [14], meaning that the $\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}$ term in the Lagrangian of $\mathcal{N} = 4$ SYM is protected from anomalous dimensions and hence so is the coupling constant g. This argument is valid to all orders in perturbation theory, which means that $\mathcal{N} = 4$ SYM is conformally invariant to all orders in perturbation theory.

2.1.2 The t'Hooft limit

Once it was discovered that QCD is the correct theory of the strong interactions it was quickly noticed that due to asymptotic freedom doing calculations in the low energy limit is very difficult. t'Hooft had an idea that the theory might simplify significantly if the number of colors was made large, i.e. $N \to \infty$. If that is the case, one could solve the theory and then do perturbation theory in terms of $1/N$. A consistent way of taking the large N limit is by keeping the quantity $\lambda \equiv g^2 N$ fixed – this is the *t'Hooft limit* and λ is the *t'Hooft coupling.* This limit can be applied to almost any gauge field theory, but say we have an SU(N) gauge theory with scalar fields in the adjoint representation. Schematically the scalar field action would look like

$$S \sim \int d^4x \, \mathrm{Tr} \left(-\partial_\mu \Phi^I \partial^\mu \Phi_I - g \, c_{IJK} \Phi^I \Phi^J \Phi^K - g^2 \, d_{IJKL} \Phi^I \Phi^J \Phi^K \Phi^L \right) \tag{2.16}$$

and indeed, ignoring the cubic term, for the $\mathcal{N} = 4$ action (2.1) this is true. We can simplify this by scaling the fields by $\tilde{\Phi}^I = g \, \Phi^I$,

$$S \sim \int d^4x \frac{1}{g^2} \, \mathrm{Tr} \left(-\partial_\mu \tilde{\Phi}^I \partial^\mu \tilde{\Phi}_I - c_{IJK} \tilde{\Phi}^I \tilde{\Phi}^J \tilde{\Phi}^K - d_{IJKL} \tilde{\Phi}^I \tilde{\Phi}^J \tilde{\Phi}^K \tilde{\Phi}^L \right) \tag{2.17}$$

thus getting an overall N/λ factor. This factor goes to infinity as we take the large N limit, but one should not forget, that the field count also goes to infinity. But what we are really interested in are Feynman diagrams and how these factors appear when evaluating them. For each vertex we get a factor of N/λ, for each propagator a factor of λ/N and for each loop an additional factor of N, since we have to sum over the color indices. Summing up, a diagram with E propagators, V vertices and L loops has a factor of

$$N^{V-E+L} \lambda^{E-V} = N^{\chi} \lambda^{E-V} = N^{2-2g} \lambda^{E-V} \tag{2.18}$$

where $\chi = 2 - 2g$ is the Euler character of the diagram and g is the genus. These quantities are better understood in terms of surfaces and indeed we can treat each Feynman diagram as a surface by using the double line notation, which uses a single directed line for a field in the fundamental representation and a reversed arrow for a field in the conjugate of the fundamental. Since the adjoint representation is roughly the fundamental times the antifundamental, fields in the adjoint are represented by double lines with arrows in the opposite directions. Two canonical examples are shown in fig. 1. These are both vacuum diagrams, but the discussion is the same for non vacuum diagrams. Now we interpret the diagrams as oriented triangulations of surfaces, which can be make compact, oriented and closed by adding points at infinity. E.g. the first diagram then corresponds to a genus zero surface, i.e. a sphere, while thee second one has genus one – it's a torus. With this identification in mind we see that the perturbation series can be reorganized as an expansion in the genus,

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{j=0}^{\infty} c_{g,j} \lambda^j = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda), \tag{2.19}$$

where f_g is some polynomial function in λ. It is now easy to see that in the large N limit all diagrams with genus higher than 0 will be suppressed and what is left are planar diagrams only, i.e. diagrams that can be drawn in a plane without any lines crossing. However the most striking feature of the genus expansion is that it reminds a perturbative string theory expansion where we also have genus expansions of the string worldsheet. This is the first hint of the correspondence between field theories and string theories. Since the argument we presented is very general and works for almost any field theory, it can be conjectured that *any* field theory has a string theory dual. And while it is only a conjecture, there are many examples of this with AdS/CFT being the first one. Obviously, different field theories would match different string theories, e.g. taking the gauge group to be SO(N) introduces non-orientable surfaces in the dual string theory, since SO(N) is a real group and there is no distinction between fundamental and anti-fundamental representations, hence there is only one possible direction for arrows. Similarly,

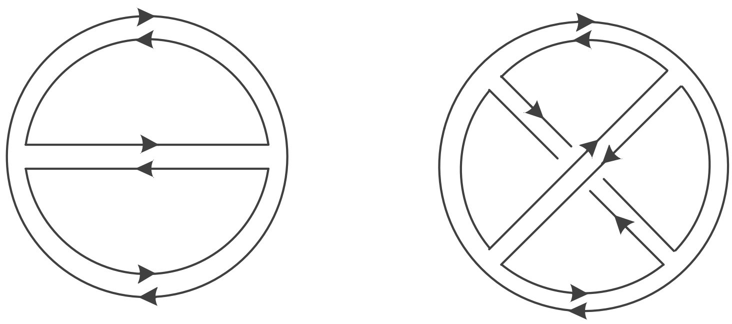


Figure 1: Canonical examples of vacuum diagrams in the double line notation, the left diagram corresponds to a sphere and the right one corresponds to a torus.

introducing fields in representations other that the adjoint, e.g. the fundamental, introduces boundaries to the surfaces. However even with this wide array of phenomenon gauge/gravity duality remains a consistent concept.

2.2 String theory and supergravity

In this section we turn our attention to the other side of the correspondence, namely type IIB string theory on an $AdS_5 \times S^5$ background. As already mentioned, taking the large N limit of a field theory makes it "string-like". The Maldacena correspondence states that for $\mathcal{N} = 4$ this is precisely the aforementioned string theory. In fact, the correspondence is stronger than that, the strongest form of the correspondence does not require taking the large N limit and states that the theories are exactly dual at *any* N. However by taking this limit we can see the correspondence more clearly, namely for the string theory this means keeping things at the classical level. Taking the low energy limit we reduce the string theory to supergravity.

We start this section with a more technical review of the less known anti de Sitter space, which plays a significant role in the correspondence. After this slight detour we introduce type IIB string theory and its low energy limit – type IIB supergravity and finish by discussing specific solutions in these theories called *branes*, which will turn out to play a big role in the correspondence.

2.2.1 Anti de Sitter space

AdS space is the Lorentzian analog of hyperbolic space, just like the Minkowski space is a Lorentzian analog of the Euclidean space. Similarly dS space is the Lorentzian analogue of a sphere, which is an example of an elliptic space. All of these spaces have constant curvature, with Minkowski space being flat, dS space having a positive constant curvature and AdS space a constant negative curvature. Hence AdS can be seen as a vacuum solution of Einstein's equations with a negative cosmological constant. A $d+1$ dimensional AdS_{d+1} space can be naturally defined as an embedding in $\mathbb{R}^{d,2}$ as

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{d} X_i^2 = -R^2 \tag{2.20}$$

which can be solved by the following parametrization [16]:

$$\begin{aligned} X_{-1} &= R\cosh\rho\sin\tau \\ X_0 &= R\cosh\rho\cos\tau \\ X_i &= R\sinh\rho\,(\Omega_{d-1})_i \end{aligned} \tag{2.21}$$

where $(\Omega_{d-1})_i$ is the collection of spherical coordinates satisfying the condition $\sum_i(\Omega_{d-1})_i^2 = 1$. E.g. for the two dimensional case these would be $\cos\alpha$ and $\sin\alpha$. The AdS_2 space together with the sphere S^2 are shown in fig. 2. These coordinates cover the whole AdS space, hence they are called global coordinates. It is worth noticing that the coordinate τ is periodic, making the topology of the space $S^1 \times \mathbb{R}^d$. Since τ is a time coordinate, this periodicity introduces closed timelike curves, which are apparent in the picture of AdS_2 in fig. 2. In order to make the space simply connected we unroll the τ coordinate, letting it take any values – this simply connected space is the universal cover of AdS, which we will have in mind from now on. The induced metric in these coordinates is

$$ds^2 = R^2(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_{d-1}^2). \tag{2.22}$$

From the embedding one may immediately note that the isometry group of the manifold is SO(2, d), which is the same as the conformal symmetry group of d dimensional Minkowski space. This relation between Minkowski and AdS spaces is the Lorentzian analogue of the fact that we can conformally compactify the Euclidean space $\mathbb{R}^n$ by adding a point at infinity thus making

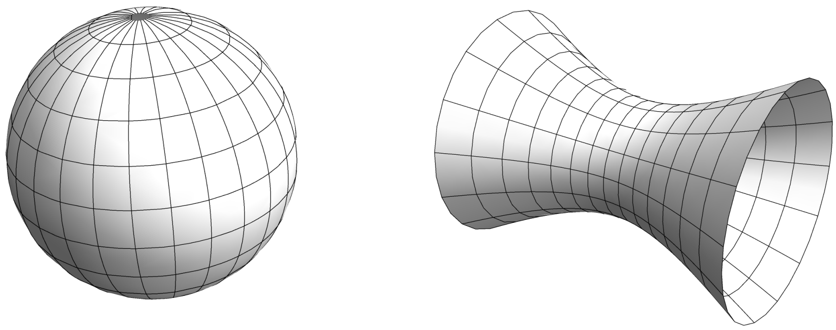


Figure 2: Images of the sphere and the AdS_2 hyperboloid.

it a sphere S^n. On the other hand, a hyperbolic space $\mathbb{H}^{n+1}$ can be conformally mapped into a disk D^{n+1} and its boundary is also the sphere S^n. We can make this statement precise for the AdS case by comparing conformal compactifications of Minkowski and AdS spaces. We start by conformally compactifying $\mathbb{R}^{1,d-1}$ with the metric

$$ds^2 = -dt^2 + dr^2 + r^2\, d\Omega_{d-2}^2 \tag{2.23}$$

first by introducing the coordinates $u_\pm = t \pm r$, which after some straightforward manipulations bring the metric to the form

$$ds^2 = -du_+ du_- + \frac{1}{4}(u_+ - u_-)^2\, d\Omega_{d-2}^2. \tag{2.24}$$

Now rescale these coordinates by $u_\pm = \tan \tilde{u}_\pm$ and introduce new time and radial coordinates $\tilde{u}_\pm = (\tau \pm \theta)/2$ bringing the metric to the form

$$ds^2 = \frac{1}{4\cos^2 \tilde{u}_+ \cos^2 \tilde{u}_-} \left(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2\right) \tag{2.25}$$

and dropping the conformal factor we are finally left with

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2. \tag{2.26}$$

Plotting the $r = \text{const}$ and $t = \text{const}$ lines in the (τ, θ) plane gives rise to the well-known triangle conformal diagram of Minkowski spacetime. We can analytically extend this triangle to the conformal space of the Einstein static universe by extending the range of the coordinates to

$$-\infty < \tau < \infty, \quad 0 \leq \theta < \pi \tag{2.27}$$

which makes the topology of this space $\mathbb{R} \times S^{d-1}$. The simplest example is the two dimensional Minkowski space, which conformally maps to $\mathbb{R} \times S^1$, i.e. a cylinder.

Now lets do the same for AdS_{d+1}, which has the metric given in (2.22). Again, introduce the rescaled coordinate θ by defining $\tan\theta = \sinh\rho$ and drop the conformal factor. This brings the metric to the form

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2, \tag{2.28}$$

which looks just like the conformally compactified metric of Minkowski (2.26). However this time the θ coordinate has the range $[0, \pi/2)$, which is only

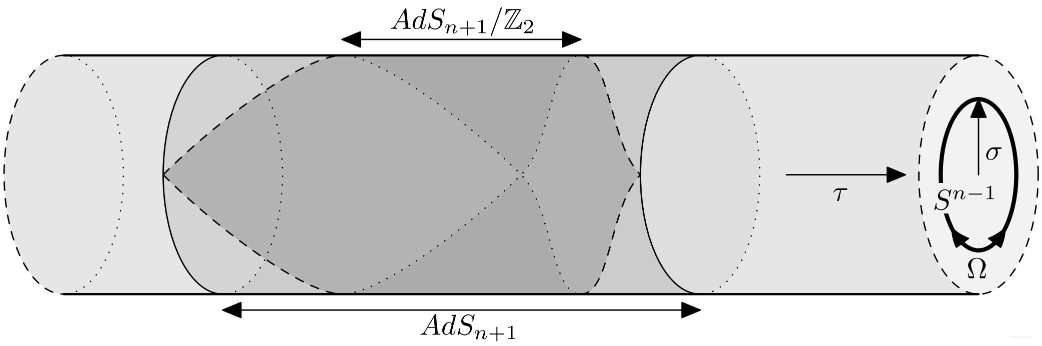


Figure 3: Anti de Sitter space: the infinite solid cylinder represents the universal cover of AdS_{n+1} whereas the medium gray region represents AdS_{n+1} with a compactified time coordinate. The dark gray region is the Poincaré patch, which covers only half of the time compactified AdS_{n+1}. The boundary of the universal cover of AdS_{n+1} is the outer region of the infinite cylinder, it's topology is $\mathbb{R} \times S^{n-1}$ [15].

half of the Minkowski case, meaning that we get only half of the Einstein static universe. This can be visualized as saying that in case of Minkowski spacetime we can take slices of time, which are spheres S^{d-1}. In this case we can also take slices of time, but the slice is only half of the sphere S^d, i.e. a hemisphere, whose boundary is S^{d-1}. Thus we see that the boundary of the conformal compactification of AdS_{d+1} is equivalent to the conformal compactification of $\mathbb{R}^{1,d-1}$. This has an important implication that AdS has a timelike boundary, meaning that in order to have a well defined physical problem on this space we need to specify boundary conditions, i.e. we can't get away with saying that fields drop off at infinity like we are used to do when dealing with Minkowski spacetime. This fact is at the heart of the AdS/CFT correspondence.

There is another set of coordinates used to label points on AdS_{d+1} called the *Poincaré coordinates*, which are given by

$$\begin{aligned} X_{-1} &= \frac{1}{2u}\left(1 + u^2(R^2 + \vec{x}^2 - t^2)\right) \\ X_0 &= R\,u\,t \\ X_i &= R\,u\,x^i \quad (i = 1, ..., d-1) \\ X_d &= \frac{1}{2u}\left(1 - u^2(R^2 - \vec{x}^2 + t^2)\right) \end{aligned} \tag{2.29}$$

and the metric in these coordinates is given by

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right). \tag{2.30}$$

The Poincaré coordinates cover only half of the AdS space, the so-called Poincaré patch – much analogous to the Rindler wedge found when using Rindler coordinates to label points in Minkowski space, since $0 \leq u < \infty$. By changing coordinates $r = 1/u$ we bring the metric to the form

$$ds^2 = \frac{1}{r^2} \left(dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right) \tag{2.31}$$

which simply means that we have a radial coordinate r with a scaled copy of Minkowski space attached to every point. The boundary of this space is Minkowski space $\mathbb{R}^{1,d-1}$. This is different from describing AdS_{d+1} with global coordinates where we found the boundary to be $\mathbb{R} \times S^{d-1}$, which can be understood by the fact the Poincaré coordinates throw out some points from the description of the manifold effectively decompactifying the sphere S^{d-1} and leaving the boundary to be $\mathbb{R}^{1,d-1}$. The Poincaré patch and it's relation to the full AdS_{d+1} and it's universal cover can be seen in fig. 3.

2.2.2 Type IIB supergravity and string theory

Historically supergravity was introduced as a candidate theory of everything. The idea here is to make supersymmetry a gauge symmetry and since we know that supersymmetry transformations are entangled with spacetime transformations, it is no surprise that we can produce gravity this way, which is basically a theory of gauged spacetime transformations. Obviously such theories would contain gauge fields as well needed to describe matter, so it seems like a good start. However nowadays supergravity theories are considered only as low energy limits of various string theories. E.g. the unique 11 dimensional SUGRA theory is considered to be the low energy limit of M-theory, which is also supposedly unique. AdS/CFT is concerned with type IIB string theory, which has type IIB supergravity as its low energy limit restricted to massless fields. At this level it can be treated as simply a theory in 10 dimensions with a lot of fields in it, which are listed in table 1. Since the theory only contains massless fields, they are classified by their representations under the little group of SO(1, 9), which is SO(8).

Due to the fact that the theory lives in 10 dimensions, the numbers of left and right-handed supercharges need not be the same [17], thus 10 dimensional

Table 1: Field content of type IIB supergravity

Field	representation	d.o.f.	name
$g_{\mu\nu}$	[2, 0, 0, 0]	35_B	graviton
$B^{(2)}_{\mu\nu}$	[0, 1, 0, 0]	28_B	B field 2-form
ϕ	[0, 0, 0, 0]	1_B	dilaton
$C^{(0)}$	[0, 0, 0, 0]	1_B	axion
$C^{(2)}_{\mu\nu}$	[0, 1, 0, 0]	28_B	R-R 2-form
$C^{(4)+}_{\mu\nu\rho\lambda}$	[0, 0, 0, 2]	35_B	self-dual 4-form
$\psi^a_{\mu\alpha}$	[1, 0, 0, 1]	112_F	Majorana-Weyl gravitinos
λ^a_α	[0, 0, 0, 1]	16_F	Majorana-Weyl dilatinos

SUGRA theories with 32 supercharges are not unique and are labeled by the doublet $\mathcal{N} = (\mathcal{N}_L, \mathcal{N}_R)$, where $\mathcal{N}_L$ and $\mathcal{N}_R$ are the numbers of left and right handed SUSY's. There are two possibilities, $(1, 1)$ and $(2, 0)$, where the former is type IIA SUGRA and the latter – type IIB. This is reflected in the fact that all fermions in IIB are left handed, i.e. the theory is chiral. Another thing to notice in the field content is that the field strength of the 4-form $C^{(4)+}$ is required to be self-dual by supersymmetry. This causes problems in writing the action for the theory, since it is very problematic to write a term for the field strength that would imply self-duality. Hence the condition $\tilde{F}_5 = *\tilde{F}_5$ is usually just written along the action as a constraint for the equations of motion. The action for the bosonic part of type IIB supergravity is given by [18]

$$\begin{aligned} S_{IIB} = &+\frac{1}{4\kappa_B^2}\int d^{10}x\,\sqrt{-g}\,e^{-2\phi}\left(2R + 8\partial_\mu\phi\partial^\mu\phi - |H_3|^2\right) \\ &-\frac{1}{4\kappa_B^2}\int d^{10}x\,\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) \\ &-\frac{1}{4\kappa_B^2}\int d^{10}x\,\sqrt{-g}\,C^{(4)+}\wedge H_3\wedge F_3 \end{aligned} \tag{2.32}$$

where the following field strength definitions were used

$$\begin{aligned} F_1 &= dC^{(0)}, \quad H_3 = dB^{(2)}, \quad F_3 = dC^{(2)}, \quad F_5 = dC^{(4)+} \\ \tilde{F}_3 &= F_3 - C^{(0)}H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}C^{(2)}\wedge H_3 + \frac{1}{2}B^{(2)}\wedge F_3. \end{aligned} \tag{2.33}$$

The field strength modulo squares are defined as

$$|F_p|^2 = \frac{1}{p!}\, g^{\mu_1\nu_1} \dots\, g^{\mu_p\nu_p}\, F^*_{\mu_1\dots\mu_p} F_{\nu_1\dots\nu_p}. \tag{2.34}$$

The action in (2.32) can be derived explicitly by taking the low energy limit of type IIB string theory. The free parameter κ_B is related to the α' parameter of string theory by $2\kappa_B^2 = (2\pi)^7(\alpha')^4$. The parameter α' determines the length of the fundamental string by $l_s = \sqrt{\alpha'}$, the tension by $T = 1/2\pi\alpha'$ and the mass by $m^2 \sim 1/\alpha'$. The string coupling constant is not really a constant, but the vacuum expectation value of the dilaton, i.e. $g_{str} = \langle e^\phi \rangle$. Thus the action (2.32) can be seen as an effective action with higher derivatives and terms in higher powers of α' and g_{str} dropped. In the full string theory one would find that perturbation theory should be done in the familiar genus expansion found in the large N limit.

2.2.3 p-branes and D-branes

Any $p+1$-form field $A^{(p+1)}$ can be associated with a $p+1$ spacetime dimensional object, since we can write a fully diffeomorphism invariant action term

$$S_{p+1} = T_{p+1} \int_{\Sigma_{p+1}} A^{(p+1)} \tag{2.35}$$

where T_{p+1} is the tension of this object. Naturally these form fields have field strengths associated by $F_{p+2} = dA^{(p+1)}$ whose fluxes are conserved – we say that the objects are charged under the form field, hence naturally their charges are conserved. Solutions of supergravity that are charged under form fields are called *p-branes*, where p is the space dimensionality of the object. Magnetic duals of p-branes in 10 dimensions can also be defined by

$$dA^{(7-p)}_{mag} = *\, dA^{(p+1)}. \tag{2.36}$$

Looking back at table 1 we expect that type IIB SUGRA should contain the so-called D(-1) branes (instantons) associated with axions and dilatons and D7 branes, which are magnetic duals of the instantons. The $B^{(2)}$ field has a string associated with it, which is usually denoted F1 and called *the fundamental string*, its magnetic dual is referred to as the NS5 brane. $C^{(2)}$ is associated with D1 strings and D5 branes and finally we have D3 branes for $C^{(4)+}$ fields, which are magnetic duals of themselves.

p-branes can be thought of as generalizations of black holes in supergravity and just like there are extremal black holes, there are extremal p-branes. Imposing such a condition on a p-brane makes it a half BPS solution, i.e. it preserves half of the supercharges. Obviously they also break the 10 dimensional Poincaré invariance down to $\mathbb{R}^{p+1} \rtimes \mathrm{SO(1,p)} \times \mathrm{SO(9-p)}$. Solutions for p-branes may be expressed explicitly in terms of a single function $H(\vec{y})$ as

$$ds^2 = H^{-1/2}(\vec{y})\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(\vec{y})d\vec{y}^2, \quad e^\phi = H(\vec{y})^{(3-p)/4}, \tag{2.37}$$

where x^μ with $\mu = 0 \dots p$ are the coordinates on the worldvolume of the p-brane and y^i with $i = 1 \dots 9-p$ are the transverse coordinates to the brane. Equations of motion imply that $H(\vec{y})$ must be a harmonic function, meaning that

$$\eta^{ij}\partial_i\partial_j H(\vec{y}) = 0. \tag{2.38}$$

The most general solution of this kind, assuming that we have maximal $\mathrm{SO(9-p)}$ symmetry in the transverse directions and spacetime is asymptotically flat as $\vec{y} \to \infty$, is given by

$$H(\vec{y}) = 1 + \frac{R^{7-p}}{|\vec{y}|^{7-p}}, \tag{2.39}$$

where R is some length scale and since α' is the only length scale in the problem it must be that $R \sim \alpha'$. The formula generalizes trivially for a multi-brane solution to

$$H(\vec{y}) = 1 + \sum_i^N \frac{C_i R^{7-p}}{|\vec{y} - \vec{y}_i|^{7-p}}. \tag{2.40}$$

It can be shown that in the case of N parallel Dp branes the coefficients C_i are given by [18]

$$C_i = g_{str} N_i (4\pi)^{(5-p)/2}\Gamma((7-p)/2)(\alpha')^{(7-p)/2}, \tag{2.41}$$

where N_i is the number of coincident branes at $\vec{y}_i$. A multi-brane solution is still a half BPS solution, which can be understood from a black hole analogy – having multiple extremal black holes put together does not affect the system in any way, since electric repulsion always cancels gravitational attraction, this is the defining property of an extremal black hole.

Since supergravity is a low energy effective theory of string theory, p-brane solutions are also solutions in string theory, where they are called *D-branes* [19] (hence the names for p-branes such as D1, D3, etc). The 'D' is for Dirichlet, which in turn comes from the fact that in string theory D-branes are objects on which open strings end, i.e. we impose Dirichlet boundary conditions for them. Like everything else in string theory, D-branes are subject to α' corrections when coming from supergravity and perturbation theory is done in terms of g_{str}. It is interesting to note that in the small coupling limit $g_{str} \rightarrow 0$, the branes become localized at spacetime and can simply be considered as defects at regions in spacetime – a freely propagating string would not feel a brane's presence until it reached the brane.

Since open strings can end on branes, they naturally describe gauge theories. This works as follows: a string excitation on a brane is equivalent to an excitation of the brane itself, i.e. its motion in the transverse directions. These excitations can be described by $9 - p$ numbers, which can be interpreted as values of scalar fields Φ^i. Since a brane is a 1/2 BPS object, these fields should be in a compatible supermultiplet of the 16 supercharges and the only possibility is the vector multiplet. Quantizing the open string thus produces an effective U(1) gauge theory [20]. The fields in the gauge theory are massless, because a string that is attached to a brane can shrink to an arbitrarily small size. If we now introduce N parallel branes, strings can attach to different branes and hence the scalar fields ${\Phi^i}_j$ have two indices. These indices must be distinguished in type IIB string theory, because strings have orientations. It is not hard to see that effectively this describes a gauge theory with spontaneous symmetry breaking, since if the branes are not coincident, i.e. they have fixed positions with respect to each other, that means that the scalar fields have vacuum expectation values. If on the other hand all branes are coincident, the symmetry group is enhanced to U(N) as can be shown by quantizing the open string. The scalar fields ${\Phi^i}_j$ can then be though of as being in the the adjoint representation of the group. Ignoring the overall position of the brane system, we are left with an SU(N) Yang-Mills gauge theory. All the fields are massless in this case, since the strings can shrink arbitrarily, but as soon as we separate any two branes, this is no longer possible – this is the Higgs mechanism at work.

2.3 The Maldacena correspondence

We now have all the necessary ingredients to present the AdS/CFT correspondence. Consider type IIB string theory with a stack of parallel D3 branes. There are two ways of looking at this system. On one hand, open strings on the N D3 branes describe $\mathcal{N} = 4$ SU(N) gauge theory, whereas the closed strings correspond to excitations of empty space, i.e. gravity. Strings can also split and join making the two subsystems interact. On the other hand, one can view the D3 branes as defects in spacetime, which curve the background geometry that closed strings move in. Closed strings far away from the branes don't feel the curvature and describe supergravity as before. Since in both cases we have free supergravity away from the branes, it is tempting to suggest that the different theories near the branes should in fact be the same. This is exactly what Maldacena did in his seminal paper [2].

In order to make the correspondence precise, one should take the low energy limit, since then the two subsystems (strings away and near the branes) decouple in both cases. First consider the $\mathcal{N} = 4$ picture, which is shown schematically in fig. 4a. The action of this system is

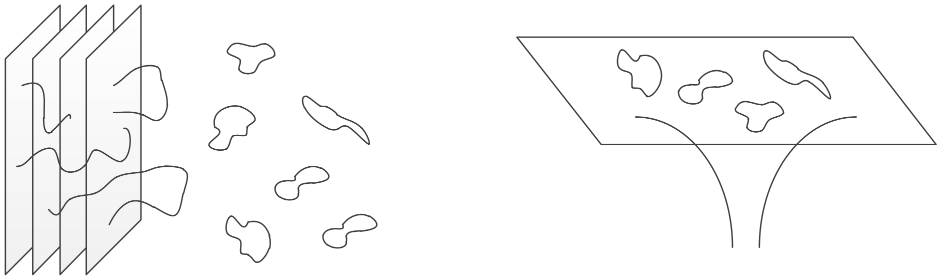
$$S = S_{brane} + S_{bulk} + S_{int}. \tag{2.42}$$

S_{brane} is the $\mathcal{N} = 4$ action (2.1) subject to α' corrections, but these can be neglected in the low energy limit $\alpha' \to 0$. This part of the action describes the brane excitations, i.e. the open strings ending on the branes. S_{bulk} is the action for excitations of empty space, i.e. closed strings in a flat 10 dimensional background. In the the low energy limit this reduces to the type IIB SUGRA action given in (2.32) with leading terms of the form

$$S_{bulk} \approx \frac{1}{2(2\pi)^7(\alpha')^4} \int d^{10}x\sqrt{-g}\, e^{-2\phi}\, 2R + \dots \tag{2.43}$$

Finally S_{int} is the interaction term, which describes string splitting and joining. Since $S_{int} \sim \alpha'$, it vanishes in the low energy limit. Thus we see that the closed and open string sectors decouple in the low energy limit.

Now consider the second picture (fig. 4b), where we take the branes to be heavy objects deforming the background geometry of spacetime. Asymptotically spacetime is still flat 10 dimensional Minkowski space, but near the branes a "throat" opens up. The geometry near the branes is described by



a) b)

Figure 4: Two ways of viewing a brane system: a) a stack of parallel D3 branes with open strings attached to the branes and closed strings floating in the background, b) the branes cause the background geometry to curve and a "throat" opens up, hence closed strings move in a highly curved background.

the metric given in (2.37), which in the case of D3 branes is

$$ds^2 = \left(1 + \frac{R^4}{|\vec{y}|^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{R^4}{|\vec{y}|^4}\right)^{1/2} d\vec{y}^2, \tag{2.44}$$

where

$$R^4 = 4\pi g_{str} N (\alpha')^2 \tag{2.45}$$

is the radius of the D3 brane. The six coordinates perpendicular to the branes $\vec{y}$ can be rewritten using polar coordinates for five of them, i.e. $d\vec{y}^2 = dy^2 + y^2 d\Omega_5^2$, so that large y corresponds to the asymptotic region far away from the branes. It is easy to see that in the limit $y \to \infty$, the metric (2.44) becomes simply $\mathbb{R}^{1,9}$. In order to study the region near the branes, we further change coordinates to $u = R^2/y$ and take the $u \to \infty$ limit. The metric now becomes

$$ds^2 = R^2 \left(\frac{1}{u^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2 \right), \tag{2.46}$$

which is the metric of $AdS_5 \times S^5$ product manifold, where both spaces have the same radius R. In the low energy limit both asymptotic backgrounds decouple and we are left with flat 10 dimensional spacetime far away from the branes and $AdS_5 \times S^5$ spacetime near the branes.

The discussion so far concerned the same physical set up, but presented it from two points of view and since both viewpoints contain closed strings in a flat $\mathbb{R}^{1,9}$ background far away from the branes, we are led to a conjecture that $\mathcal{N} = 4$ SU(N) super Yang-Mills and type IIB string theory on $AdS_5 \times S^5$

must be *dual theories*, meaning that there should be a direct correspondence between all degrees of freedom and all physical observables in these theories. The first sign that the correspondence indeed holds is the fact that the symmetries of both theories match, since $\mathcal{N} = 4$ SYM is symmetric under the conformal group SO$(2,4)$, which is the isometry group of AdS_5 and R-symmetry SO(6), which is the isometry group of S^5. The 16 supercharges on a D3 brane are enhanced to 32, because AdS_5 is a maximally supersymmetric space, so the numbers of supercharges also match. In fact, type IIB string theory on $AdS_5 \times S^5$ is also symmetric under the full symmetry group of $\mathcal{N} = 4$ SYM, which is PSU$(2,2|4)$.

2.3.1 Parameter matching and limits

If both theories in the correspondence are to be identified, we should be able to derive relations among the parameters describing the theories. $\mathcal{N} = 4$ SYM is parametrized by the number of colors N and the coupling constant g (or the t'Hooft coupling $\lambda = g^2 N$). Type IIB strings on $AdS_5 \times S^5$ are parametrized by the radius of both of the product spaces R, the number of D3 branes N, the string coupling constant g_{str} and the slope parameter α'. The most obvious identification is that the number of D3 branes N is equal to the number of colors in $\mathcal{N} = 4$ SYM. This is also the flux of the 5-form RR field strength F_5 over the 5-sphere S^5, i.e.

$$\int_{S^5} * F_5 = N. \tag{2.47}$$

We already saw that the AdS_5 and S^5 radius satisfies a nontrivial relation

$$R^4 = 4\pi g_{str} N (\alpha')^2. \tag{2.48}$$

Finally, comparing the actions of both theories suggests a further identification $g^2 \sim g_{str}$, however from the physics of D-branes we find a more complicated relation [21]

$$\frac{4\pi i}{g^2} + \frac{\theta}{2\pi} = \frac{i}{g_{str}} + \frac{\chi}{2\pi}, \tag{2.49}$$

where θ is the instanton angle of $\mathcal{N} = 4$ SYM and χ is the expectation value of the axion scalar field $C^{(0)}$ from the type IIB SUGRA multiplet. This makes sense, since g_{str} is related to the expectation value of the dilaton, i.e.

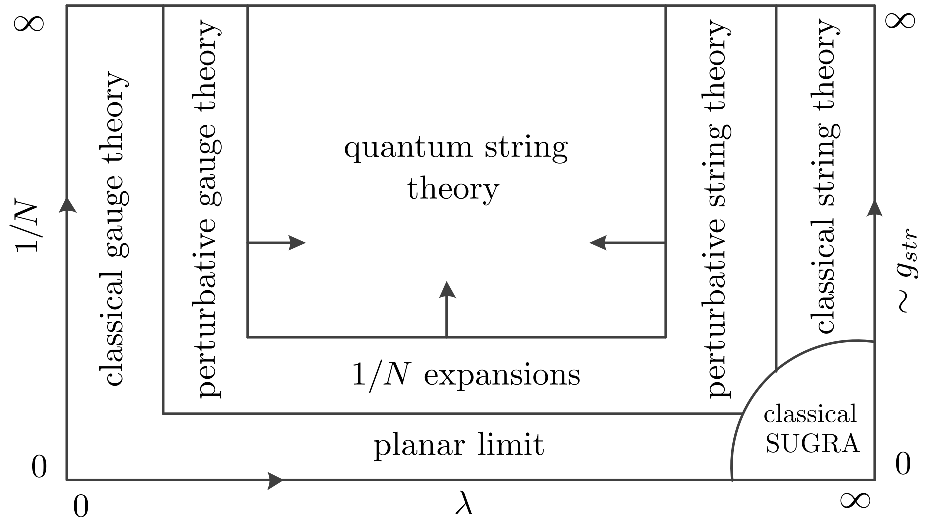


Figure 5: The parameter space for the AdS/CFT correspondence and the various possible limits and expansions that can be used to approach the full quantum string theory.

$g_{str} = \left\langle e^{\phi} \right\rangle$, which is the only other scalar field in the theory. If the instanton angle is set to zero, we are left with $4\pi g_{str} = g^2$ and hence

$$\frac{R^2}{\alpha'} = \sqrt{\lambda}, \tag{2.50}$$

which expresses the strong weak duality, i.e. the low energy supergravity limit of $\alpha' \to 0$ corresponds to the strong coupling regime $\lambda \to \infty$ of $\mathcal{N} = 4$ SYM and vice versa. The reasoning for the Maldacena correspondence up to now relied on taking the low energy limit, i.e. $\alpha' \to 0$. It is then natural to ask where exactly does the correspondence hold ? One should start this discussion by first of all noting that the correspondence is only a conjecture, i.e. there is no rigorous proof for the correspondence at any limit, even though there is a lot of evidence for it. The strongest form of the correspondence states that it holds for all values of all of the parameters, i.e. at any N and g_{str}. It is highly nontrivial to prove or even check such a statement, since that would involve doing calculations in a fully quantum string theory on a curved background and there is no way to do it at the moment. A weaker form of the correspondence states that it holds at least in the large N or planar limit, i.e. taking $N \to \infty$ and keeping the t'Hooft coupling λ fixed. Roughly this means that we are left with classical string theory, since

$g_{str} \to 0$, which is also what we see on the gauge theory side when taking the t'Hooft limit. This is the regime where integrability techniques work and they have been used extensively to check the correspondence in this limit and so far the correspondence seems to hold. Finally the weakest form of the correspondence involves taking a further limit of sending $\alpha' \to 0$ on top of the t'Hooft limit. Looking back at (2.50) we see that this corresponds to making the radius of the curvature very large, meaning that the background becomes flat and we are left with classical supergravity. Summing up we see that α' in effect controls the shift between classical and quantum, whereas increasing g_{str} turns on string interactions. The parameter space is illustrated in fig. 5.

2.3.2 The dynamical statement

The fact that the symmetries of the theories on both sides of the correspondence match is not enough to justify it. In order for it to be a true correspondence we have to give a dynamical statement describing how states/operators in one theory map to the other theory. Since we are dealing with quantum field theories we seek a bijective map between the generating functionals of the theories, however if we restrict to the low energy limit of string theory, where we still have hope to solve something, the mapping should be between classical fields in supergravity and quantum operators in gauge theory. Classical fields in 10 dimensional supergravity are usually dimensionally reduced on S^5 to give Kaluza-Klein towers of modes in AdS_5 with Minkowski space as the boundary, the so-called 5 dimensional *bulk fields* $\varphi(x^\mu, z)$. Very roughly the ansatz is that each bulk field maps to an operator $\mathcal{O}$ in the 4 dimensional gauge theory and the exact correspondence is given by [22]

$$Z_{SYM}[\varphi_0] = \int \mathcal{D}\varphi \, e^{iS_{AdS}[\varphi]|_{\varphi_0}}, \tag{2.51}$$

where $S_{AdS}[\varphi]$ is the classical action for the 5 dimensional supergravity on AdS_5 and the path integral is done over all φ field configurations with values of φ_0 on the boundary of AdS_5. Z_{SYM} is the generating functional for the operator $\mathcal{O}$ as a function of the source φ_0 for the operator, i.e.

$$Z_{SYM}[\varphi_0] \equiv \int \mathcal{D}\varphi \, e^{i\int d^4x(\mathcal{L}_{SYM}+\varphi_0\mathcal{O})}. \tag{2.52}$$

Thus we see that every field configuration on AdS_5 perturbs the gauge theory on the boundary by adding a term $\varphi_0 \, \mathcal{O}$ to the lagrangian, hence each classical field configuration corresponds to a different gauge theory on the boundary. Once the generating functional is known it is trivial to calculate correlation functions

$$\langle \mathcal{O}(x)\mathcal{O}(y)\ldots\mathcal{O}(z)\rangle = \frac{\delta}{\delta\varphi_0(x)}\frac{\delta}{\delta\varphi_0(y)}\cdots\frac{\delta}{\delta\varphi_0(z)}Z[\varphi_0]|_{\varphi_0=0} \tag{2.53}$$

From here on the idea is simple, in order to calculate correlation functions one has to first solve the classical wave equation in AdS_5 for the 5 dimensional fields and then calculate using standard quantum field theory methods. We won't go into details of these calculations, since they are beyond the scope of this paper, but it is worth noting that it is indeed possible to derive correlation functions using supergravity methods that agree with results from pure $\mathcal{N}=4$ SYM calculations, e.g. one can show that the two-point function in coordinate space is given by [16]

$$\left\langle \mathcal{O}(x)\tilde{\mathcal{O}}(y)\right\rangle = \frac{(2\Delta-4)\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}\frac{1}{|x-y|^{2\Delta}}, \tag{2.54}$$

where Δ is the classical dimension of the operator $\mathcal{O}$. This agrees with the well known result from conformal field theory, which states that two-point functions are highly constrained and up to a constant factor they are given by

$$\left\langle \mathcal{O}(x)\tilde{\mathcal{O}}(y)\right\rangle \sim \frac{1}{|x-y|^{2\Delta}}. \tag{2.55}$$

Three-point correlation functions show a similar agreement. Of course, this simply confirms the fact that both theories have matching symmetries.

3 Integrability in AdS/CFT

In this section we focus on the integrable structures found in $\mathcal{N} = 4$ SYM and type IIB superstring theories. Roughly speaking, a theory is said to be integrable when it has an infinite amount of conserved charges, meaning that the symmetry is so restrictive that in some sense everything is related to everything by symmetry, hence the theory can be solved exactly [8]. At first sight it may seem that such a symmetric theory would be trivial, but in this case, even though everything is rather restricted, the theories are far from trivial. Formally the type of symmetry encountered in integrable systems can often be implemented by a *quantum algebra*, i.e. a deformed universal enveloping algebra of an affine Lie algebra [23]. This leads to the study of *Yangians* as the formal objects behind integrability [24].

Since $\mathcal{N} = 4$ SYM and type IIB superstrings are related by the AdS/CFT duality, integrability provides a way to check the correspondence, since if one theory is integrable, so must be the other. Integrability then allows one to calculate various observables at any coupling, thus bypassing the strong/weak duality problem, i.e. one can compare calculations at the same coupling in both theories and even interpolate between them, something that is not possible using perturbation theory. Even though this does not prove the correspondence, it is definitely a step in the right direction.

In this section we introduce integrability by discussing the spectral problem in AdS/CFT, which concerns with finding the spectra of states in both theories. We start from the $\mathcal{N} = 4$ SYM side by showing how the dilatation operator for single trace local operators at one-loop level can be related to spin chains and hence shown to be integrable. We then proceed with showing how the spin chain model can be solved exactly using the Bethe ansatz and how this procedure generalizes to the full theory and to all loops. The same problem is discussed from the string theory side where it is also found to be solvable exactly by the method of spectral curves. We show how this solution also emerges from gauge theory in the asymptotic limit confirming that indeed the theories are related. We finish the section by discussing further advances in other problems in AdS/CFT which also benefit from integrability and the limits where integrability is thought to break down.

3.1 Integrability in gauge theory

3.1.1 The spectral problem

Integrability in $\mathcal{N} = 4$ SYM was first discovered while trying to find the spectrum of the theory, i.e. the conformal dimensions of various operators. We already saw that in any conformal field theory the two-point correlation functions are very restricted at tree level. For operators that are eigenvalues of dilatations, the correlators have the same form at all loop levels,

$$\left\langle \mathcal{O}(x)\, \tilde{\mathcal{O}}(y) \right\rangle \approx \frac{1}{|x-y|^{2\Delta(g)}}, \tag{3.1}$$

where $\Delta(g)$ is the dimension of the operator. Classically $\Delta = \Delta_0$ is simply the mass dimension, but at the quantum level it receives radiative corrections and acquires an *anomalous dimension* γ, such that $\Delta(g) = \Delta_0 + \gamma(g)$, where the anomalous dimension depends on the coupling. Usually the corrections are small and the correlator can be expanded as

$$\left\langle \mathcal{O}(x)\, \tilde{\mathcal{O}}(y) \right\rangle \approx \frac{1}{|x-y|^{2\Delta_0}} \left(1 - \gamma \ln \Lambda^2 |x-y|^2\right). \tag{3.2}$$

Obviously we want to calculate the spectrum for operators that are gauge invariant and since all fields are in the adjoint representation of the gauge group, all gauge invariant operators will consist of traces over the color indices. In general such an operator has the form of

$$\begin{aligned} \mathcal{O}_{i_1\mu i_2\alpha\ldots i_n\ldots j_1\nu\beta\ldots j_n}(x) &= \mathrm{Tr}\left[\Phi_{i_1}(x)\mathcal{D}_\mu\Phi_{i_2}(x)\psi_\alpha(x)\ldots\Phi_{i_n}(x)\right]\times\ldots \\ &\quad \cdots\times \mathrm{Tr}\left[\Phi_{j_1}(x)\mathcal{D}_\nu\psi_\beta(x)\ldots\Phi_{j_n}(x)\right]. \end{aligned} \tag{3.3}$$

We assume that all the fields are evaluated at the same spacetime point x making the operators local. In the planar limit we can restrict ourselves to single trace operators, since states with multiple traces always involve non-planar Feynman diagrams, which are suppressed in the planar limit. To see how non-planar diagrams emerge and get suppressed consider the chiral primary operator

$$\Psi = \mathrm{Tr}[Z^L] = Z^a{}_b Z^b{}_c \ldots Z^l{}_a, \tag{3.4}$$

where the scalar field Z and its conjugate $\tilde{Z}$ have the standard tree level correlator

$$\left\langle Z^a{}_b(x) \tilde{Z}^{b'}{}_{a'}(y) \right\rangle_{\text{tree}} \approx \frac{\delta^a{}_{a'}\delta_b{}^{b'}}{|x-y|^2}. \tag{3.5}$$

In order to find the anomalous dimension of the chiral primary operator Ψ we must calculate $< \Psi(x)\tilde{\Psi}(x) >$. We do this by using Wick's theorem and plugging in the two-point correlator (3.5), which produces a lot of terms with delta function contractions between the adjoint indices. Some examples are

$$\ldots \delta^{a'}{}_{a}\, \delta^{a}{}_{a'}\, \delta^{b'}{}_{b}\, \delta^{b}{}_{b'}\, \delta^{c'}{}_{c}\, \delta^{c}{}_{c'}\ \ldots \tag{3.6a}$$

$$\ldots \delta^{a'}{}_{c}\, \delta^{c}{}_{a'}\, \delta^{b'}{}_{a}\, \delta^{a}{}_{b'}\, \delta^{c'}{}_{b}\, \delta^{b}{}_{c'}\ \ldots \tag{3.6b}$$

$$\ldots \delta^{a'}{}_{a}\, \delta^{a}{}_{b'}\, \delta^{c'}{}_{b}\, \delta^{b}{}_{a'}\, \delta^{b'}{}_{c}\, \delta^{c}{}_{c'}\ \ldots \tag{3.6c}$$

These contractions have a graphical interpretation. Consider the scalar field $Z^{a}{}_{b}$ as a dot and each contraction of the adjoint indices as a line connecting these dots, then the chiral primary operator Ψ is simply a circle. Wick's theorem says that in order to find the correlator $< \Psi(x)\tilde{\Psi}(x) >$ we must sum all possible ways we can connect the dots in the circle of Ψ to the dots in the circle of $\tilde{\Psi}$. All the delta function contractions that we get after expanding the correlator represent precisely all the possible ways we can contract the dots in the circles. The three excerpts of contractions shown in (3.6) can be represented graphically as shown in fig. 6. One can immediately notice that the first two are planar, while the third one is intersecting itself. Evaluating the three contractions we immediately see that planar ones produce a factor of N^3 while the non-planar one produces a factor of N, i.e. non-planar diagrams are suppressed and we can discard them once we take the planar limit $N \to \infty$. All that's left then are cyclic permutations of lines by shifting all of them as seen in fig. 6 while going from (a) to (b). There are $L - 1$ shifts that can be done in this way, since after making a full circle we return to the initial configuration. Thus finally for the chiral primary correlator at

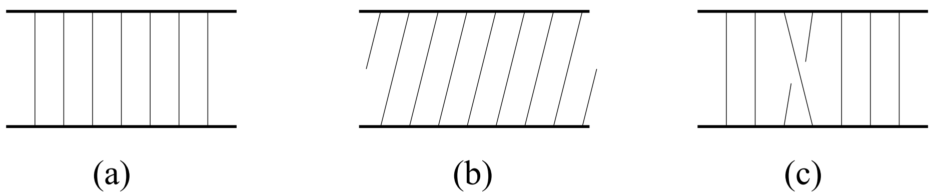


Figure 6: Possible types of contractions between fields in traces of operators, which are represented by horizontal lines. Vertical lines represent the contractions. (a) and (b) are two examples of planar contractions while (c) is an example of a non-planar contraction.

tree level we find

$$\left\langle \Psi(x)\,\tilde{\Psi}(y)\right\rangle_{\text{tree}} \approx \frac{LN^L}{|x-y|^{2L}}, \tag{3.7}$$

where N^L comes from the contractions and L from all the possible planar ways we can contract. This can easily be generalized for correlators of operators with arbitrary scalar fields $\Phi_{I_1 I_2 \dots I_L}(x) = \mathrm{Tr}\,[\Phi_{I_1}(x)\Phi_{I_2}(x)\dots\Phi_{I_L}(x)]$ to

$$\left\langle \Phi_{I_1 I_2 \dots I_L}(x)\,\tilde{\Phi}^{J_1 J_2 \dots J_L}(y)\right\rangle_{\text{tree}} \approx \frac{1}{|x-y|^{2L}}\left(\delta_{I_1}^{J_1}\delta_{I_2}^{J_2}\dots\delta_{I_L}^{J_L} + \text{cycles}\right), \tag{3.8}$$

where "cycles" refers to terms with the J indices pushed. I and J are flavor indices, the color indices are suppressed.

So far so good, but in order to calculate anomalous dimensions we have to go beyond tree level. This may seen like a highly nontrivial thing to do, since we expect not only scalar interactions, but also gluon exchanges and fermion loops appearing. Luckily the symmetry of the theory allows one to calculate all gluon and fermion effects in one go. First let's concentrate on the bosonic sector of the theory ignoring gluons. The action (2.1) contains a single scalar-only interaction term[†]

$$\begin{aligned} S_\Phi &= -\frac{g^2}{4}\sum_{I,J}\int d^4x\ \mathrm{Tr}\,[\Phi_I,\Phi_J][\Phi_I,\Phi_J] \\ &= -\frac{g^2}{4}\sum_{I,J}\int d^4x\ \left(\mathrm{Tr}\,[\Phi_I\Phi_I\Phi_J\Phi_J] - \mathrm{Tr}\,[\Phi_I\Phi_J\Phi_I\Phi_J]\right). \end{aligned} \tag{3.9}$$

In order to calculate the correlator (3.8) at one-loop level, one should insert this term and Wick contract. Just like in tree level, we only have to keep planar diagrams. For the interaction terms this means that only neighboring fields can interact. This drastically reduces the number of terms we get after Wick contracting. Because of that it is enough to consider a length two operator $\Phi_{I_k I_{k+1}}$ and with a bit of work one can show that at one-loop level we get [14]

$$\begin{aligned} \left\langle \Phi_{I_k I_{k+1}}(x)\,\tilde{\Phi}^{J_k J_{k+1}}\right\rangle_{\text{one-loop}} &= \tfrac{\lambda}{16\pi^2}\tfrac{\ln(\Lambda^2|x-y|^2)}{|x-y|^{2L}} \times \\ \times \left(2\delta_{I_k}{}^{J_{k+1}}\delta_{I_{k+1}}{}^{J_k} - \delta_{I_k I_{k+1}}\delta^{J_k J_{k+1}} - \delta_{I_k}{}^{J_k}\delta_{I_{k+1}}{}^{J_{k+1}}\right), \end{aligned} \tag{3.10}$$

[†]A careful reader might notice that we raise and lower the I and J indices at will, but that's not a problem, because they are SO(6) indices and there is no distinction between upper and lower.

where $\lambda = g^2 N$ is the t'Hooft coupling. Comparing this to (3.8) we see that effectively the interactions permute and contract the delta function indices. We can introduce exchange and trace operators to make this explicit. The permutation operator, also called the exchange operator, $\mathcal{P}_{l,l+1}$ is defined by it's action on a set of delta functions as

$$\mathcal{P}_{l,l+1}\, \delta_{I_1}{}^{J_1} \ldots \delta_{I_l}{}^{J_l} \delta_{I_{l+1}}{}^{J_{l+1}} \ldots \delta_{I_L}{}^{J_L} = \delta_{I_1}{}^{J_1} \ldots \delta_{I_l}{}^{J_{l+1}} \delta_{I_{l+1}}{}^{J_l} \ldots \delta_{I_L}{}^{J_L} \tag{3.11}$$

and the trace operator $\mathcal{K}_{l,l+1}$ is defined as

$$\mathcal{K}_{l,l+1}\, \delta_{I_1}{}^{J_1} \ldots \delta_{I_l}{}^{J_l} \delta_{I_{l+1}}{}^{J_{l+1}} \ldots \delta_{I_L}{}^{J_L} = \delta_{I_1}{}^{J_1} \ldots \delta_{I_l I_{l+1}} \delta^{J_l J_{l+1}} \ldots \delta_{I_L}{}^{J_L}. \tag{3.12}$$

Using these operators we can rewrite the correlator in (3.10) in a more compact notation

$$\begin{aligned} &\left\langle \Phi_{I_k I_{k+1}}(x)\, \tilde{\Phi}^{J_k J_{k+1}} \right\rangle_{\text{one-loop}} = \\ &\qquad = \frac{\lambda}{16\pi^2} \frac{\ln(\Lambda^2 |x-y|^2)}{|x-y|^{2L}} \quad \left(2\, \mathcal{P}_{k,k+1} - \mathcal{K}_{k,k+1} - 1\right) \delta_{I_k}^{J_k} \delta_{I_{k+1}}^{J_{k+1}}. \end{aligned} \tag{3.13}$$

This result includes four scalar interactions only, however as mentioned before at one-loop level we can also have gluon interactions and fermion loops in scalar propagators. The nice thing about these is that such interactions don't alter the flavor index structure, i.e. there are no permutations or traces. Basically this happens because the gluon transforms trivially under R-symmetry and hence can't change the flavor index (which transforms under R-symmetry). Fermions on the other hand do transform under R-symmetry and it is a miracle that happens only at one-loop level that they don't alter the flavor structure. Thus all of these interactions contribute a constant term C, which we can determine later. We can generalize our one-loop result with all interactions included for operators of arbitrary length,

$$\begin{aligned} \left\langle \Phi_{I_1 I_2 \ldots I_L}(x)\, \tilde{\Phi}^{J_1 J_2 \ldots J_L}(y) \right\rangle_{\text{one-loop}} &= \frac{\lambda}{16\pi^2} \frac{\ln(\Lambda^2 |x-y|^2)}{|x-y|^{2L}} \times \\ \times \sum_{l=1}^{L} \left(2\, \mathcal{P}_{l,l+1} - \mathcal{K}_{l,l+1} - 1 + C\right) &\left(\delta_{I_1}^{J_1} \delta_{I_2}^{J_2} \ldots \delta_{I_L}^{J_L} + \text{cycles}\right). \end{aligned}$$

Combining this with the tree level result (3.8) and comparing to the general expression of a two-point function at one-loop level (3.2) we can deduce the anomalous dimension γ, which now becomes an operator Γ because of the flavor mixing. It is given by

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(-2\,\mathcal{P}_{l,l+1} + \mathcal{K}_{l,l+1} + 1 - C\right). \tag{3.14}$$

At first sight it may seem strange that what was supposed to be a number, i.e. a correction to the mass dimension of an operator has turned out to be an operator acting on the flavor space, i.e. a matrix. But this is very natural and in fact expected, since interactions can change the flavor of fields and we can't be sure that an operator at the quantum level has the same flavor indices as it does at the classical level. This line of thinking may lead to a natural question, why do we have mixing between the scalars only and not between all the fields in the theory including fermions, which miraculously do not appear. It turns out that this is a one-loop feature only and mixing becomes a problem at higher loop levels. In fact, the next subsection about closed sectors is devoted to the question of operator mixing.

Now that we have acknowledged that the anomalous dimension is a matrix and found an expression for it, the next logical step would be diagonalizing it and finding the flavor eigenstates. One example of such an eigenstate is the chiral primary operator Ψ. Since it contains scalar fields of only one type, the permutation and trace operators act trivially on it. Thus we see that

$$\Gamma\,\Psi = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(-2 + 1 - C\right)\Psi, \tag{3.15}$$

but by definition a chiral primary has an anomalous dimension of zero, which then fixes the constant C to -1. And finally we get

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(2 - 2\,\mathcal{P}_{l,l+1} + \mathcal{K}_{l,l+1}\right). \tag{3.16}$$

A keen eye might already notice that this expression resembles a Hamiltonian of a spin chain. In fact, this is hardly surprising, since from the very beginning we were talking about fields as points in some closed line, which indeed resembles a spin chain. Furthermore the correlators that we were calculating are nothing more that propagators from one state of the chain to another, hence no wonder that the operator describing this evolution looks like a Hamiltonian for a spin chain. This identification is very useful, because the spin chains that appear in AdS/CFT are integrable and can be solved exactly, which gives us hope that we can apply the same techniques here and solve the spectral problem in $\mathcal{N} = 4$ exactly. This is indeed what was first done in the seminal paper [9], which launched the integrability program in AdS/CFT. However saying that the spectral problem can be solved exactly in this particular case is too strong, since we are only at one-loop level. Nevertheless it turns out that one can apply the same techniques going beyond one-loop level. What happens is that long range interactions start appearing in the Hamiltonian as one goes higher in loops. What is more unexpected is that under certain limits one can actually guess how the solution should look like at all loops. Further techniques like the thermodynamic Bethe ansatz can then be applied to solve the spectral problem exactly without assuming any limits. The coming sections will explore these techniques in more detail.

3.1.2 Closed sectors

In the previous section we showed that the anomalous dimension operator is in fact a matrix, signaling that there is operator mixing in the theory. However we saw that at least at one loop level, the scalar fields seem to mix only among themselves, suggesting that there might by closed mixing sectors. It is not hard to see that this is indeed the case, since the dilatation operator commutes with the Lorentz and R-symmetry generators, thus it preserves Lorentz and R-symmetry charges of the operators in question. What is more, the dilatation operators at each loop level, including D^0, the bare dimension operator, commute among themselves [14], which means that only operators with the same bare dimensions can form closed sectors. Summing up, we characterize operators by six charges - $[\Delta, S_1, S_2; J_1, J_2, J_3]$, where Δ is the bare dimension, S_1 and S_2 are the Lorentz charges and J_1, J_2 and J_3 are the three SO(6) R-symmetry charges. Closed sectors then consist of operators having the same charges.

The prime example of a closed sector is the SU(2) sector, which consists of local single trace operators with M complex scalar fields of one type and $L - M$ complex scalars of another type, e.g.

$$\Psi_{\mathrm{SU}(2)} = \mathrm{Tr}\left(ZZWZW\ldots ZW\right), \tag{3.17}$$

where

$$Z = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) \quad \text{and} \quad W = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4). \tag{3.18}$$

Such operators have charges $[L, 0, 0; L - M, M, 0]$ and there is no other way to combine other operators (except permuting the W and Z operators inside the trace) to get this set of charges, thus they form a closed sector. The name SU(2) comes from the fact that any two scalar fields make up a doublet under the $SU(2) \subset SO(6)$ subgroup of R-symmetry. Traces of three scalar fields (W, Z and X) don't form a closed SU(3) sector, since they can also mix with fermions. E.g. consider the operator $\mathrm{Tr}\,(XWZ)$, which has charges $[3, 0, 0; 1, 1, 1]$ – the same charges can be produced by combining two fermions with charges $[3/2, 1/2, 0; 1/2, 1/2, 1/2]$ and $[3/2, -1/2, 0; 1/2, 1/2, 1/2]$. Since there is no other way to produce these charges apart from introducing two fermions, these fields form another closed sector called the $SU(2|3)$ sector [25]. Other closed sectors include $SU(1|1)$, $SU(1|2)$ (see [26] for details) and $SL(2) \simeq SU(1, 1)$ which even appears in QCD [27].

There are also sectors which are closed only at one-loop level, the prime example being the SO(6) sector that we encountered in the previous section when introducing spin chains. It consists of all real scalar fields Φ_I. Unlike in the SU(3) case, there is no way to form a closed sector at higher loop levels here without resorting to the full symmetry group $PSU(2, 2|4)$. The reason why this sector is closed at one loop level is that mixing outside of it is *dynamical*, i.e. operators can mix to other operators with different numbers of fields in the trace and it turns out that this can happen starting at two-loop level only [25], leaving the sector closed at one loop.

3.1.3 Spin chains and the Bethe ansatz

Let us now focus on the SU(2) sector of $\mathcal{N} = 4$ SYM. The anomalous dimension operator is then given by

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} (1 - \mathcal{P}_{l,l+1}) \,. \tag{3.19}$$

Up to a constant factor this is the same as the Hamiltonian for the Heisenberg spin chain (also called the XXX spin chain), which is a quantum description of a one dimensional magnet. The Hamiltonian is given by

$$\mathbf{H} = \sum_{l=1}^{L} (1 - \mathcal{P}_{l,l+1}) \,, \tag{3.20}$$

which can also be rewritten in terms of Pauli matrices as

$$\mathbf{H} = 2\sum_{l=1}^{L}\left(\frac{1}{4} - \vec{S}_l \cdot \vec{S}_{l+1}\right), \quad \vec{S}_l = \frac{1}{2}\vec{\sigma}_l. \tag{3.21}$$

Hence solving the spectral problem in $\mathcal{N} = 4$ SYM translates into solving the Schrödinger equation

$$\mathbf{H}\,|\psi\rangle = E\,|\psi\rangle, \tag{3.22}$$

where we now seek to find the energy eigenvalues for the Hamiltonian of the spin chain. If the chain is short, this is a trivial diagonalization problem that can be easily solved by a present day computer. However this problem was first solved analytically by Hans Bethe in a time when computers were still in their infancy. The original solution (English translation available in [28]) now goes by the name of *coordinate Bethe ansatz* and it is by far one of the most important and beautiful solutions in physics in the past century, which is still very widely used even to this day. The idea is to make an educated guess for the wave function $|\psi\rangle$, plug it in to the Schrödinger equation and determine when does it actually hold. This produces a set of algebraic Bethe ansatz equations for a set of variables unimaginatively called the Bethe roots. All observables can then be expressed in terms of these numbers as simple algebraic functions, thus transforming a diagonalization problem to an algebraic problem. This has an enormous advantage, since in the asymptotic limit, when the spin chain is very large, instead of diagonalizing an infinite matrix, the set of algebraic equations actually simplify and produce integral equations, which can be solved.

We already discussed that the anomalous dimension Γ must be a matrix due to operator mixing in $\mathcal{N} = 4$ SYM. In the SU(2) sector we have mixing only between two scalar fields, e.g. W and Z. In the spin chain picture these fields can be treated as up and down spin states, i.e.

$$|\uparrow\rangle = Z = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle = W = \begin{pmatrix}0\\1\end{pmatrix}, \tag{3.23}$$

thus local single trace operators can be treated as states of a spin chain, e.g.

$$\mathrm{Tr}\,(WWZWWZW) = |\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle \tag{3.24}$$

Due to the cyclicity of the trace all rotations of the chain are equivalent. We should also specify the periodicity boundary condition

$$\vec{S}_{L+1} = \vec{S}_1. \tag{3.25}$$

The operators $\vec{S}_l$ act as Pauli matrices on the l'th spin site and trivially on all the others. Since a spin "chain" with a single site would have a state space $\mathbb{C}^2$, a spin chain of length L has a state space $\mathbb{C}^{\otimes_L 2}$, which has 2^L basis vectors and the Hamiltonian is then a $2^L \times 2^L$ matrix, which we need to diagonalize. Working directly with Pauli matrices one can find some simple results directly, e.g. it is trivial to show that the chiral primary operator

$$|\Psi\rangle = \mathrm{Tr}\left[Z^L\right] = |\uparrow\uparrow \ldots \uparrow\rangle \tag{3.26}$$

is an eigenstate of the Hamiltonian with zero energy, i.e. it is the ferromagnetic ground state of the spin chain, which we will denote as $|0\rangle$ from now on. This is expected, since we know that chiral primaries have zero anomalous dimensions. Another eigenstate of the Hamiltonian is the *single magnon* state, defined as

$$|p\rangle = \sum_{n=1}^{L} e^{ipn}|n\rangle, \tag{3.27}$$

where $|n\rangle$ is the ground state with the n'th spin flipped,

$$|n\rangle = S_n^- \,|0\rangle = |\uparrow\uparrow\uparrow \ldots \downarrow \ldots \uparrow\uparrow\uparrow\rangle, \tag{3.28}$$

here p is formally just a parameter, but it can be interpreted as the momentum of the excitation travelling in the spin chain. Due to the cyclicity of the chain the momentum is quantized,

$$p = \frac{2\pi}{L}n, \quad n \in \mathbb{Z}, \tag{3.29}$$

where n is the mode number. The energy of the excitation is given by the dispersion relation

$$E(p) = 4\sin^2\frac{p}{2}. \tag{3.30}$$

Now consider a two magnon state

$$|\psi\rangle = \sum_{n<m} \psi(n,m)\,|n,m\rangle, \quad |n,m\rangle = S_n^-\,S_m^-\,|0\rangle. \tag{3.31}$$

The situation is not so trivial this time, since the two magnons might scatter among themselves. We now plug this into (3.22) and find the conditions for $\psi(n,m)$, which are

$$\begin{aligned} E\,\psi(n,m) = 4\,\psi(n,m) - \psi(n+1,m) - \psi(n-1,m) \\ - \psi(n,m+1) - \psi(n,m-1) \end{aligned} \tag{3.32}$$

when $m > n + 1$ and

$$E\,\psi(n, n+1) = 2\,\psi(n, n+1) - \psi(n-1, n+1) - \psi(n, n+2) \qquad (3.33)$$

when $m = n + 1$, i.e. when the two magnons scatter. The solution is now a superposition of single magnon states

$$\psi(n, m) = e^{ikn+ipm} + S(k, p)e^{ipn+ikm}, \qquad (3.34)$$

where

$$S(p, k) = \frac{\frac{1}{2}\cot\frac{k}{2} - \frac{1}{2}\cot\frac{p}{2} - i}{\frac{1}{2}\cot\frac{k}{2} - \frac{1}{2}\cot\frac{p}{2} + i} \qquad (3.35)$$

is the scattering matrix. As required, such a state is an eigenstate and the energy is given by

$$E = E(p) + E(k), \qquad (3.36)$$

i.e. it is simply the sum of the single magnon energies. Finally the spin chain periodicity condition imposes the following equations

$$e^{ikL}\, S(p, k) = e^{ipL}\, S(k, p) = 1. \qquad (3.37)$$

It is now straightforward to generalize this procedure, which is exactly what Bethe did in his seminal paper [28]. The wave function for M spins down can be written as

$$|\psi\rangle = \sum_{1\le l_1<l_2<\cdots<l_M\le L} \psi(l_1, l_2, \ldots, l_M)\, S^-_{l_1}\, S^-_{l_2}\, \ldots\, S^-_{l_M}\, |0\rangle. \qquad (3.38)$$

The sum is chosen in a way so as not to over count states. The Bethe ansatz is the educated guess of the wave function

$$\psi(l_1, l_2, \ldots, l_M) = \sum_{\sigma\,\in\, perm(1,2,\ldots\, M)} A(p)\, e^{ip_{\sigma_1} l_1 + ip_{\sigma_2} l_2 + \cdots + ip_{\sigma_M} l_M}, \qquad (3.39)$$

where the sum runs over all permutations of the down spin labels $1, 2, \ldots, M$. p_i are the momenta of the down spins, which can be treated as excitations moving in the vacuum state of the spin chain. The ansatz then looks like a superposition of plane waves. As in the two magnon case, one should now plug in the ansatz and find the conditions that make it work. The result is a set of algebraic equations, called the *Bethe equations* [29]

$$e^{ip_k L} = -\prod_{\substack{j=1\\ j\neq k}}^{M} \frac{e^{ip_j} - e^{ip_k} + 1}{e^{ip_k} - e^{ip_j} + 1} \quad \text{for } k = 1, 2, \ldots, M \qquad (3.40)$$

and the amplitude is given by

$$A(r) = \text{sign}(\sigma) \prod_{j<k} \left(e^{ip_j} - e^{ip_k} + 1 \right) . \tag{3.41}$$

These equations can be interpreted physically once rewritten as

$$e^{ip_k L} \prod_{\substack{j=1 \\ j\neq k}}^{M} S(p_j, p_k) = 1, \quad \text{where } S(p_j, p_k) = -\frac{e^{ip_k} - e^{ip_j} + 1}{e^{ip_j} - e^{ip_k} + 1}. \tag{3.42}$$

This is simply saying that if we take a magnon, carry it around the spin chain, the total phase change which is a result of free propagation ($e^{ip_k L}$) and scattering with other magnons ($S(p_j, p_k)$) must be trivial. Changing variables to

$$e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}, \quad u_k = \frac{1}{2} \cot \frac{p_k}{2}, \tag{3.43}$$

brings the Bethe equations (3.40) to a more familiar form

$$\left(\frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \tag{3.44}$$

where now one solves for the Bethe roots u_k, also known as magnon rapidities. It is now straightforward to see that this general solution reproduces the two magnon scenario we discussed earlier. The energy of the M magnon state is given by

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}, \tag{3.45}$$

which also agrees with the single and two magnon examples.

They key thing worth noting in (3.42) is that the spin chain can be fully described in terms of the scattering matrix for just two particles, i.e. the full M particle scattering matrix factorizes. This is the defining property of integrability [30], since factorized scattering means that individual momenta are conserved in each two particle scattering producing a tower of conserved quantities – just the thing one would want in an integrable system.

3.1.4 Beyond one-loop level

The next step in solving the spectral problem is increasing the loop level. For the SU(2) sector this has first been done for two-loops using mainly diagrammatic methods and by fixing the structure of the operator by symmetry.

The resulting dilatation operator is given by [31]

$$\Gamma_{2-loop} = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} \left(-4 + 6\,\mathcal{P}_{l,l+1} - (\mathcal{P}_{l,l+1}\,\mathcal{P}_{l+1,l+2} + \mathcal{P}_{l+1,l+2}\,\mathcal{P}_{l,l+1})\right). \quad (3.46)$$

In the spin chain picture this corresponds to a Hamiltonian for a long range spin chain with two nearest neighbour interactions. This spin chain has been shown to be integrable [31]. This result has been extended to three, four and five loops [13]. The explicit expressions for the dilatation operator at higher loops get more and more lengthy and complicated, but a pattern emerges that at loop level l the dilatation operator can be identified with a Hamiltonian of a long range spin chain where at most l nearest neighbours in the chain interact. What is even more remarkable is that these spin chains also turn out to be integrable [32], which hints that integrability may be an all loop phenomenon. This was in part verified by solving the spectral problem in the asymptotic limit, i.e. when the spin chain length L becomes infinite, but the number of excitations M is kept finite. The solution is given by conjecturing a set of *asymptotic Bethe ansatz equations*, which since their original inception have been extensively verified [33, 34]. The equations have the same form as in the one-loop case (3.42), but the scattering function for two magnons gets modified to [30]

$$S(p_i, p_j) = \frac{u(p_i) - u(p_j) + i}{u(p_i) - u(p_j) - i} \times S_D(p_i, p_j), \quad (3.47)$$

where $S_D(p_i, p_j)$ is the so called dressing factor (an explicit expression for it can be found in [35]) and the rapidities are now defined as

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}. \quad (3.48)$$

The outcome is that all one-loop results get slightly modified, e.g. the magnon dispersion relation (3.30) becomes

$$E(p) = \frac{8\pi^2}{\lambda} \left(\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} - 1 \right), \quad (3.49)$$

which in the low coupling limit $\lambda \to 0$ agrees with the one-loop result as it should. For many magnon states the energy is still given by the sum of individual magnon energies. It is truly remarkable that such a simple solution exists even though the Hamiltonian of the all-loop spin chain is not known. But even though such an easy generalization to an all-loop solution looks promising, it is only the first step towards the full solution of the spectral problem in $\mathcal{N} = 4$ SYM.

3.1.5 Full solution to the spectral problem

The next step in generalizing the SU(2) solution is to include bigger sectors with the hope of eventually solving the spectral problem for the full PSU(2, 2|4) theory. The Bethe ansatz was in fact originally suggested for the SO(6) sector at one-loop level (recall that SO(6) is not closed beyond one-loop level). This result was extended to other sectors, e.g SU(2|3) in [25], PSU(1, 1|2) in [36] until finally the ansatz was generalized to a spin chain with any symmetry group G where each spin site lives in some representation of the group R_i [37]. The technique for arriving at this result is the so called *algebraic Bethe ansatz*. It is a more formal version of the coordinate Bethe ansatz, which is based on a very physical picture. The algebraic Bethe ansatz on the other hand is more in the spirit of integrability and hence is very formal, for an excellent introduction see [38]. Thus for an arbitrary symmetry group the Bethe equations are given by [39]

$$\left(\frac{u_{i,k} + \frac{i}{2}V_k}{u_{i,k} - \frac{i}{2}V_k}\right)^L = \prod_{l=1}^{r}\prod_{\substack{j=1\\ j\neq i}}^{J_l} \frac{u_{i,k} - u_{j,l} + \frac{i}{2}M_{kl}}{u_{i,k} - u_{j,l} - \frac{i}{2}M_{kl}}, \tag{3.50}$$

where M_{kl} is the Cartan matrix of the symmetry group and V_k is the vector of highest weights for the representation that the spin sites live in. This is a set of equations for the Bethe roots $u_{k,i}$, where $k = 1, \ldots, \mathrm{rank}(G)$ and $i = 1, \ldots, J_k$ with J_k being the number of excitations of type k (each type corresponds to a different node of the Dynkin diagram, hence k has $\mathrm{rank}(G)$ possible values). The total number of excitations is then $J = \sum J_k$. All of the conserved charges of the system can now be given in terms of the Bethe roots as [39]

$$Q_r = \frac{i}{r-1}\sum_{l=1}^{r}\sum_{j=1}^{J_r}\left(\frac{1}{\left(u_{j,l} + \frac{i}{2}V_l\right)^{r-1}} - \frac{1}{\left(u_{j,l} - \frac{i}{2}V_l\right)^{r-1}}\right). \tag{3.51}$$

In particular the energy is simply the second conserved charge, i.e.

$$E = Q_2 = \frac{i}{r-1}\sum_{l=1}^{r}\sum_{j=1}^{J_r}\left(\frac{1}{u_{j,l} + \frac{i}{2}V_l} - \frac{1}{u_{j,l} - \frac{i}{2}V_l}\right). \tag{3.52}$$

It is now straightforward to specify the solution for the full theory. PSU$(2,2|4)$ is a supergroup of rank 7 with the Cartan matrix given by

$$M = \begin{pmatrix} -2 & +1 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -2 \end{pmatrix} \tag{3.53}$$

and the highest weights for the representation $\mathbf{4}|\mathbf{4}$ are all 0 except for $V_4 = 1$. Thus the the one-loop Bethe equations for the full theory are given by

$$\begin{aligned}
1 &= \prod_{\substack{j=1\\j\neq k}}^{K_1} \frac{u_{1,k}-u_{1,j}-i}{u_{1,k}-u_{1,j}+i} \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}} \\
1 &= \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}-\frac{i}{2}}{u_{2,k}-u_{3,j}+\frac{i}{2}} \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}-\frac{i}{2}}{u_{3,k}-u_{2,j}+\frac{i}{2}} \prod_{\substack{j=1\\j\neq k}}^{K_3} \frac{u_{3,k}-u_{3,j}+i}{u_{3,k}-u_{3,j}-i} \prod_{j=1}^{K_4} \frac{u_{3,k}-u_{4,j}-\frac{i}{2}}{u_{3,k}-u_{4,j}+\frac{i}{2}} \\
\left(\frac{u_{4,j}+\frac{i}{2}}{u_{4,j}-\frac{i}{2}}\right)^L &= \prod_{j=1}^{K_3} \frac{u_{4,k}-u_{3,j}-\frac{i}{2}}{u_{4,k}-u_{3,j}+\frac{i}{2}} \prod_{\substack{j=1\\j\neq k}}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{K_5} \frac{u_{4,k}-u_{5,j}-\frac{i}{2}}{u_{4,k}-u_{5,j}+\frac{i}{2}} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}-\frac{i}{2}}{u_{5,k}-u_{6,j}+\frac{i}{2}} \prod_{\substack{j=1\\j\neq k}}^{K_5} \frac{u_{5,k}-u_{5,j}+i}{u_{5,k}-u_{5,j}-i} \prod_{j=1}^{K_4} \frac{u_{5,k}-u_{4,j}-\frac{i}{2}}{u_{5,k}-u_{4,j}+\frac{i}{2}} \\
1 &= \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}-\frac{i}{2}}{u_{6,k}-u_{5,j}+\frac{i}{2}} \\
1 &= \prod_{\substack{j=1\\j\neq k}}^{K_7} \frac{u_{7,k}-u_{7,j}-i}{u_{7,k}-u_{7,j}+i} \prod_{j=1}^{K_2} \frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}}.
\end{aligned} \tag{3.54}$$

These equations also have to be supplemented with the condition that the total momentum in the spin chain must be a multiple of 2π due to the cyclicity of the chain,

$$1 = e^{iP} = \prod_{j=1}^{K_4} \frac{u_{k,j} + \frac{i}{2}}{u_{k,j} - \frac{i}{2}}. \tag{3.55}$$

Generalizing this solution even further to higher loops is not so trivial. However as we already saw in the SU(2) sector, the asymptotic all-loop solution might actually be very simple. Indeed that turned out to be the case for the full theory too and the asymptotic all-loop Bethe equations for the full theory have been conjectured [40]. They are given by

$$\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k}\, x_{4,j}^+}{1 - 1/x_{1,k}\, x_{4,j}^-} \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \\
1 &= \left(\frac{x_{4,k}^-}{x_{4,k}^-}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k}x_{4,j}^+}{1 - 1/x_{1,k}x_{4,j}^-}\, \sigma^2(x_{4,k}, x_{4,j}) \times \\
&\times \prod_{j=1}^{K_1} \frac{1 - 1/x_{4,k}^-\, x_{1,j}}{1 - 1/x_{4,k}^+\, x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - 1/x_{4,k}^-\, x_{7,j}}{1 - 1/x_{4,k}^+\, x_{7,j}} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-} \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}}{u_{6,k} - u_{5,j} - \frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + \frac{i}{2}}{u_{6,k} - u_{7,j} - \frac{i}{2}} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}}{u_{7,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{7,k}\, x_{4,j}^+}{1 - 1/x_{7,k}\, x_{4,j}^-},
\end{aligned} \tag{3.56}$$

where $\sigma^2(x_{4,k}, x_{4,j})$ is the dressing factor that we already encountered in the asymptotic SU(2) solution, it's full expression can be found in [39]. The deformation variables x are defined by

$$x + \frac{1}{x} = \frac{4\pi}{\sqrt{\lambda}} u, \qquad x^\pm + \frac{1}{x^\pm} = \frac{4\pi}{\sqrt{\lambda}} \left(u \pm \frac{i}{2}\right). \tag{3.57}$$

Finally we have the momentum periodicity condition

$$1 = e^{iP} = e^{i(p_1+\cdots+p_{K_4})} = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-}. \tag{3.58}$$

One can now construct all conserved charges of the spin chain for the Bethe roots in an algebraic fashion, e.g. the energy is given by

$$E = \frac{i\sqrt{\lambda}}{2\pi} \sum_k \left(\frac{1}{x_{4,k}^+} - \frac{1}{x_{4,k}^-} \right). \tag{3.59}$$

The final step to the truly *full* solution of the spectral problem in $\mathcal{N} = 4$ SYM is going beyond the asymptotic limit. If one wants to find energy levels of finite length spin chains, one has to introduce finite size corrections, which are also called *wrapping corrections*. The name comes from the fact that at higher loops the interactions in the spin chain become long ranged and in the asymptotic limit they become infinitely ranged. Hence making the spin chain length finite means that these interactions wrap around the chain and in effect produce non-local interactions, which have to be accounted for. The first attempt in including finite size corrections was inspired by a solution to a similar problem in field theory [41], where the effects of finite volume were evaluated for the mass spectrum of a field theory. The same idea was applied to the string theory picture of spin chains thus producing the so called Lüscher corrections, which consist of perturbative formulas for including finite size effects in the spectral problem [42]. Extensive tests at strong coupling have verified that the corrections match perturbative string theory results [43, 44] and checks at weak coupling show a match to diagrammatic gauge theory calculations [45].

Recently a different method has been proposed in [46] for solving the spectral problem in finite size exactly, which is based on the *thermodynamic Bethe ansatz*. The Lüscher corrections are then simply exponential expansions in $1/L$ of the full solution. The idea here is to consider the mirror theory of the Euclidean version of the original theory, where by Euclidean we mean a theory that is defined by the analytical continuation in the $y = it$ complex time variable. The mirror theory is then defined by exchanging time and size coordinates with $x = i\tau$ being the complex version of the mirror time coordinate τ and y being the space coordinate. The mirror theory is obviously different from the original, e.g. for the asymptotic SU(2) spin chain the original dispersion relation (3.49) gets mirror inverted to [26]

$$\tilde{E}(\tilde{p}) = 2 \arcsin \frac{\sqrt{\tilde{p}\,(16\,\pi^2 + \tilde{p}\,\lambda)}}{8\pi}, \tag{3.60}$$

the scattering matrix also has a different pole structure, meaning that the theory has different bound states from the original ones. But the remarkable thing is that mirroring a theory preserves integrability, meaning that one can solve it with an asymptotic Bethe ansatz [47]. One can use this fact, since the partition functions for these theories satisfy the obvious identity

$$Z(L, R) = \tilde{Z}(R, L), \tag{3.61}$$

where L is the length scale of the original theory and R is the time scale. At asymptotic time scales the partition function is dominated by contributions from the ground state, this applies to any length scale of the system, thus in the asymptotic time limit

$$Z(L, R) = \mathrm{Tr}\, e^{-R\,H(L)} \xrightarrow[R \to \infty]{} e^{-R\,E_0(L)}. \tag{3.62}$$

This limit corresponds to the infinite length limit for the mirror model, which we can solve using the asymptotic Bethe ansatz, thus

$$\tilde{Z}(R, L) = \mathrm{Tr}\, e^{-L\,\tilde{H}(R)} \xrightarrow[L \to \infty]{} \sum_n e^{-L\,\tilde{E}_n(R)}, \tag{3.63}$$

where $\tilde{H}$ is the Hamiltonian of the mirror theory. Now we simply identify the partition functions and solve for the ground state energy in the original theory at any length L. The result is then a simple integral given by [47]

$$E_0(L) = -\frac{1}{2\pi} \sum_r \int du\,(\partial_u \tilde{p}) \log\left(1 + e^{-\epsilon_r(u)}\right), \tag{3.64}$$

where $\epsilon_r(u)$ is the so called pseudo energy, defined in terms of the density of solutions with some charge r in the mirror theory and we sum over all charges that describe the solutions. Energies of excited states at finite length can then be found by analytic continuation [48].

The argument for the finite size solution presented here is very sketchy and for the full PSU$(2, 2|4)$ theory the story is obviously way more complicated, but everything that we discussed has indeed been done for the full theory. The solution is written in terms of a Y-system [46], which is a set of algebraic equations frequently found in integrable systems [49]. The Y-system equations have been verified on numerous occasions and have thus far passed every test [50].

3.2 Integrability in string theory

Since we found that the $\mathcal{N} = 4$ super Yang-Mills gauge theory is integrable, the AdS/CFT correspondence suggests that type IIB string theory on $AdS_5 \times S^5$ should also be integrable and all the concepts we found in gauge theory, like spin chains, conformal dimensions and closed sectors of the theory should somehow translate to analogous concepts in string theory. And indeed they do! It's not hard to guess that spin chain states translate to solutions of strings and the conformal dimensions of corresponding operators for spin chain states are simply the energies of the corresponding string solutions. Hence the spectral problem in string theory corresponds to finding energy levels of strings. String theory also has closed sectors, which are submanifolds of the $AdS_5 \times S^5$ target manifold where strings move, e.g. spin chains in the SU(2) sector translate to strings moving in a $S^3 \times \mathbb{R}$ submanifold. Most importantly the underlying string theory is integrable as expected [10], meaning that there is an infinite tower of conserved charges, which can be used to solve the theory exactly.

3.2.1 Strings in $AdS_5 \times S^5$

Type IIB string theory on any curved background is defined by the Green-Schwarz action [51], which has the bosonic part of

$$S_B = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu, \tag{3.65}$$

where $X^\mu(\sigma, \tau)$ is the string embedding map into the target space, which has the metric $G_{\mu\nu}$ and h^{ab} is the string worldsheet metric. In the low energy limit this action reduces to the type IIB supergravity action given in (2.32). Strings on $AdS_5 \times S^5$ are described by a coset space sigma model [7] with the target superspace of

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1) \times \mathrm{SO}(5)}, \tag{3.66}$$

which has the bosonic part of $AdS_5 \times S^5$. The action can also be reformulated in terms of the algebra current

$$J = -g^{-1}\mathrm{d}g \in \mathfrak{psu}(2,2|4), \tag{3.67}$$

where $g(\sigma, \tau) \in \mathrm{PSU}(2,2|4)$ is the map from the string worldsheet to the supergroup $\mathrm{PSU}(2,2|4)$. Since the target space is the coset of $\mathrm{PSU}(2,2|4)$ by

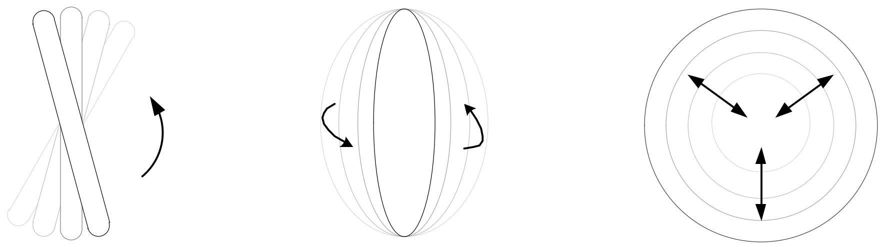


Figure 7: Some examples of classical string solutions in AdS_5, from the left: rotating folded string, rotating circular string and a pulsating string.

SO(4, 1) × SO(5), the map g has an extra gauge symmetry

$$g \to gH, \quad H \in \mathrm{SO}(4,1) \times \mathrm{SO}(5) \tag{3.68}$$

and the supercurrent transforms as

$$J \to H^{-1} J H \pm H^{-1} \mathrm{d}H \tag{3.69}$$

It can also be decomposed as

$$J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} \tag{3.70}$$

under the $\mathbb{Z}_2$ grading of PSU(2, 2|4). The Green-Schwarz action (3.65) for the string in $AdS_5 \times S^5$ can now be written as [52]

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{STr}\left(J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)} + \Lambda \wedge J^{(2)} \right), \tag{3.71}$$

where Λ is a Lagrange multiplier, which ensures that $J^{(2)}$ is supertraceless. A supertrace $\mathrm{STr}\, M$ for an element of a supergroup is defined as

$$\mathrm{STr}\, M = \mathrm{STr} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \mathrm{Tr}\, A - \mathrm{Tr}\, D. \tag{3.72}$$

Here the supergroup element M is written as a matrix with the $\mathbb{Z}_2$ grading made manifest, A and D are the bosonic parts.

Once the action is in place, all that's left to do is to derive the equations of motion and solve them. Unfortunately that is easier said that done, in fact, currently there is no way to solve the full quantum string theory on a curved background, which makes the action highly non-linear. However since the background is highly symmetric there are various limits one can take and hope that the problem simplifies tremendously [53]. A popular approach is to look for classical string solutions in $AdS_5 \times S^5$, which can be shown to very well approximate quantum solutions under certain limits. The machinery of finding classical solutions is very involved (see [54]) and the solutions involve complicated elliptic and hyperelliptic functions. Some examples of solutions are conceptually shown in fig. 7, these include rotating circular strings, rotating folded strings and many more.

One of the simplest examples of a string solution is the so called BMN ground state of the string [30]. It is a solution restricted to the $S^3 \times \mathbb{R}$ submanifold of the target space, where the $\mathbb{R}$ factor corresponds to the global time direction of AdS_5. Since a string is restricted to a S^3 spacelike manifold, it carries two charges under the isometry group of this space, these correspond to two of the three charges of S^5, which we denoted as $[J_1, J_2, J_3]$, we can choose the two charges to be J_1 and J_3. The charge under $\mathbb{R}$ is obviously the string energy E. Thus we see that such a string is labeled by the same charges as a spin chain state in the SU(2) sector of $\mathcal{N} = 4$ SYM, where states are characterized by the number of W and X fields in the trace, which correspond to J_1 and J_3, and by the anomalous dimension Γ, which corresponds to the energy for the string. The action for such a string is given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \, \partial_a \vec{X} \cdot \partial^a \vec{X} + \Lambda \left(|\vec{X}|^2 - 1 \right), \tag{3.73}$$

which is simply the Nambu-Goto action for a string on a 3-sphere, hence the Lagrange multiplier. Here $\vec{X}(\sigma, \tau)$ is the embedding map from the worldsheet to the 4 dimensional target space, which satisfies $|\vec{X}|^2 = 1$. We can now introduce complex coordinates

$$Z_1 = X_1 + iX_2, \quad Z_2 = X_3 + iX_4. \tag{3.74}$$

The BMN ground state solution is then simply

$$Z_1 = e^{i\kappa\tau}, \quad Z_2 = 0, \tag{3.75}$$

which has charges $E = J_1 = \sqrt{\lambda}\,\kappa$ and $J_3 = 0$. We immediately identify this solution with the chiral primary operator $\mathrm{Tr}[Z^L]$ in $\mathcal{N} = 4$ SYM or the ferromagnetic ground state in the spin chain picture. In the string picture this solution corresponds to a point-like massless string moving at the speed of light around the equator of S^3 [30].

One could go further and find classical solutions corresponding to magnon states in the spin chain picture, which indeed can be done – magnons turn out to be described by the so called *giant magnon* solutions in string theory [55]. These are closely related to sine-Gordon models, which are known to be integrable, hence once again confirming the expectation that string theory on $AdS_5 \times S^5$ should be integrable (at least on the classical level). However since we are trying to solve the spectral problem, i.e. find the energy levels of the string solutions, we should remind ourselves that in the spin chain picture we were able to do it without actually finding explicit solutions of the spin chain states. This is a hint that there should be a way of doing the same in string theory, i.e. finding the energy levels of strings without actually solving the equations of motion.

3.2.2 Integrability and the spectral curve

In the previous section we defined strings on $AdS_5 \times S^5$ in terms of the algebra current J, given in (3.67). This current has the property of being flat,

$$dJ - J \wedge J = 0, \tag{3.76}$$

and what is more, one can even define a one parameter family of connections from it by [56]

$$\begin{aligned} L(x) = J^{(0)} + \frac{x^2+1}{x^2-1}\, J^{(2)} - \frac{2x}{x^2-1}\left(*J^{(0)} - \Lambda\right) \\ + \sqrt{\frac{x+1}{x-1}}\, J^{(1)} + \sqrt{\frac{x-1}{x+1}}\, J^{(3)}, \end{aligned} \tag{3.77}$$

which are flat for all x,

$$dL(x) - L(x) \wedge L(x) = 0. \tag{3.78}$$

Here $L(x)$ is the *Lax connection* and x is the spectral parameter. The existence of such a set of connections signals that the theory is at least classically integrable. This can be shown by constructing the monodromy matrix

$$\Omega(x) = \mathcal{P} \exp \oint_\gamma L(x), \tag{3.79}$$

where γ is any path wrapping the worldsheet cylinder. Since the connection is flat, by definition it is path independent and we can evaluate the integral along any $\tau = const$ loop. Furthermore, shifting the τ value corresponds to doing a similarity transformation on the monodromy matrix [57], meaning that the eigenvalues must be time independent. Thus we have an infinite tower of conserved charges, hinting that the theory may be integrable. Denote the eigenvalues of the monodromy matrix as

$$\{e^{i\hat{p}_1(x)},\ e^{i\hat{p}_2(x)},\ e^{i\hat{p}_3(x)},\ e^{i\hat{p}_4(x)} \mid e^{i\tilde{p}_1(x)},\ e^{i\tilde{p}_2(x)},\ e^{i\tilde{p}_3(x)},\ e^{i\tilde{p}_4(x)}\}. \tag{3.80}$$

Here the quantities $p(x)$ are called *quasi-momenta*. The bar as always denotes the $\mathbb{Z}_2$ grading and we use the convention that hatted quantities correspond to AdS_5 variables and quantities with tildes correspond to S^5. From elementary algebraic geometry we know that the zeroes of any polynomial define an algebraic curve and since the eigenvalues $e^{ip(x)}$ are the zeroes of the characteristic polynomial of the monodromy matrix, they must also define an algebraic curve. A key idea in the development of integrability was the realization that this algebraic curve also known as the *spectral curve* can be used to define classical string solutions [58]. It is highly nontrivial to reconstruct a classical string solution given a set of quasi-momenta, yet they provide a very convenient way of describing solutions and they are very useful for solving the spectral problem. In that sense the spectral curve is the string analogue of the Bethe equations, which can also be used to find explicit solutions, but the their true power lies in their ability to efficiently solve the spectral problem.

The characteristic equation for the monodromy matrix is of order eight, meaning that the algebraic curves it defines can be thought of as cuts connecting eight sheets of a Riemann surface. A cut connecting sheets i and j is denoted as $\mathcal{C}^{ij}$ and the quasi-momenta on these sheets have discontinuities

$$p_i(x + i\epsilon) - p_j(x - i\epsilon) = 2\pi n_{ij}, \tag{3.81}$$

where n_{ij} is an integer. Four of the eight sheets correspond to the AdS_5 part of the string target space and the other four to the S^5 part, hence the indices i and j take on values

$$i \in \{\,\tilde{1},\, \tilde{2},\, \hat{1},\, \hat{2}\,\}, \quad j \in \{\,\tilde{3},\, \tilde{4},\, \hat{3},\, \hat{4}\,\} \tag{3.82}$$

and we define p to have either a hat or a tilde based on the index, i.e.

$$p_{\hat{i}}(x) \equiv \hat{p}_i(x) \quad \text{and} \quad p_{\tilde{i}}(x) \equiv \tilde{p}_i(x). \tag{3.83}$$

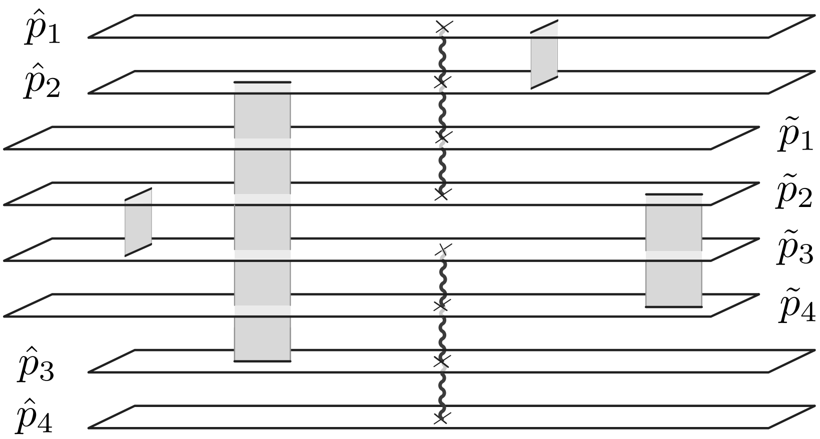


Figure 8: Examples of cuts connecting the eight sheets of the Riemann surface corresponding to the spectral curve for strings in $AdS_5 \times S^5$. The wavy line corresponds to the pole at $x = 1$.

We determine the polarization of a solution by the type of sheets the corresponding cut connects, e.g. if it connects two hatted sheets, the string is polarized in the AdS_5 part of the background and if it connects mixed sheets it is a fermionic excitation. Solutions in closed sectors, e.g. strings moving in the $\mathbb{R} \times S^3$ submanifold of the target space will be limited to cuts between a subset of the eight sheets. Some examples of cuts are shown in fig. 8. For each cut we associate the so called *filling fraction* defined by

$$S_{ij} = \pm \frac{\lambda}{8\pi^2 i} \oint_{\mathcal{C}^{ij}} \left(1 - \frac{1}{x^2}\right) p_i(x) dx, \tag{3.84}$$

where a plus sign is used for indices with a hat and a minus for indices with a tilde. These are the action angle variables for the theory, which is another concept from classical integrability [59]. Roughly they measure the length of the cut, it is also known that they correspond to the excitation numbers of strings or the number of Bethe roots in the spin chain picture [58], hence they are integers.

Since the Lax connection has poles at $x = \pm 1$, so do the quasi-momenta. Due to the Virasoro constraint, which comes about from the diffeomorphism invariance of the worldsheet, the residues of the quasi-momenta are constrained to

$$\{\hat{p}_1,\ \hat{p}_2,\ \hat{p}_3,\ \hat{p}_4 \,|\, \tilde{p}_1,\ \tilde{p}_2,\ \tilde{p}_3,\ \tilde{p}_4\} = \frac{\{\,\alpha_\pm,\ \alpha_\pm,\ \beta_\pm,\ \beta_\pm,\ |\,\alpha_\pm,\ \alpha_\pm,\ \beta_\pm,\ \beta_\pm\}}{x \pm 1}. \tag{3.85}$$

An additional constraint on the quasi-momenta comes from the fact that the algebra $\mathfrak{psu}(2,2|4)$ has an automorphism, which is the cause for an additional $\mathbb{Z}_4$ grading. The constraints are given by [56]

$$\begin{aligned} \tilde{p}_{1,2}(x) &= -\tilde{p}_{2,1}(1/x) - 2\pi m \\ \tilde{p}_{3,4}(x) &= -\tilde{p}_{4,3}(1/x) - 2\pi m \\ \hat{p}_{1,2,3,4}(x) &= -\hat{p}_{2,1,4,3}(1/x). \end{aligned} \tag{3.86}$$

These relations define an inversion symmetry. Finally one can look at the asymptotics of the quasi-momenta as the spectral parameter becomes infinite. In this limit the Lax connection becomes related to the Noether currents of the theory and hence one can relate the quasi-momenta to the charges of the global symmetry algebra by [53]

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} = \frac{2\pi}{x} \begin{pmatrix} +\mathcal{E} - \mathcal{S}_1 + \mathcal{S}_2 \\ +\mathcal{E} + \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} + \mathcal{S}_1 + \mathcal{S}_2 \\ \hline +\mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_3 \\ +\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 \end{pmatrix}, \tag{3.87}$$

where the charges are rescaled by $\mathcal{Q} = Q/\sqrt{\lambda}$. Thus we see that we can characterize the quasi-momenta by describing their behaviour at poles and under symmetries, by their asymptotics and their filling fractions.

Let us now revisit the simplest string solution we know, the BMN string and describe it using the spectral curve. Not surprisingly it is the simplest algebraic curve possible, containing no poles or cuts except the trivial ones at $x = \pm 1$. The quasi-momenta are given by [60]

$$\tilde{p}_{1,2} = -\tilde{p}_{3,4} = \hat{p}_{1,2} = -\hat{p}_{3,4} = \frac{2\pi \mathcal{J} x}{x^2 - 1}. \tag{3.88}$$

From the asymptotic behaviour as $x \to \infty$ we can determine the charges of this solution by comparing to (3.87) and we find

$$\mathcal{J}_1 = \mathcal{J}, \qquad \mathcal{E} = \kappa = \mathcal{J}, \tag{3.89}$$

all other charges being zero. Many other solutions can be characterized this way, e.g. the giant magnon corresponds to a two cut solution [57].

As already mentioned, the key advantage of describing solutions using algebraic curves is the ability to solve the spectral problem without actually solving the equations of motion. We already saw this at the classical level, where finding the energy of a solution amounts to looking at the asymptotic behaviour of the quasi-momenta. Quasi-classical analysis of the spectral curve enables one to go beyond the classical theory and find quantum corrections to the energy levels of classical solutions. The idea of quasi-classical analysis and the spectral curve for that matter traces back to the early days of quantum mechanics. Consider a particle in a smooth one dimensional potential described by the wave function $\psi(x)$. Define the quasi-momentum by

$$p(x) \equiv \frac{\hbar}{i}\frac{\psi'(x)}{\psi(x)}, \tag{3.90}$$

the Schrödinger equation then looks like

$$p^2(x) - i\hbar\, p'(x) = 2m(E - V), \tag{3.91}$$

which would be the classic energy momentum relation if it were not for the $\hbar$ term. The quasi-momentum has a pole for each zero of the wave function, so for a highly excited state this will be some big number $N \to \infty$ and we would recover the classical solution. What is more, the poles get closer and closer to each other and in the classical limit they condense to form a cut connecting two sheets in a Riemann surface. Thus in the classical limit we recover the spectral curve of this system. We also know that the number of poles is given by

$$\frac{1}{2\pi\hbar}\oint_{\mathcal{C}} p(x)\, dx = N, \tag{3.92}$$

which is also the Bohr-Sommerfield quantization condition. This integral effectively measures the size of the cut when the poles condense to a cut, thus this is the filling fraction. This simplified discussion illustrates how one could go from a classical system to a quantum one. The idea of quasi-classical analysis is to start with a classical solution and perturb it by adding microscopic cuts to the Riemann surface, which effectively describe some quantum excitations. This is exactly what has been done for various string solutions in $AdS_5 \times S^5$ [57]. One can then proceed with comparing the spectra of string solutions beyond the classical level with spectra of spin chain states at higher loop levels and the results so far have been encouraging [61].

3.2.3 Spectral curve from gauge theory

An interesting question is how the spectral curve arises from gauge theory, since because of the AdS/CFT correspondence we know that everything we do in string theory must have an analogue in gauge theory. The answer is almost obvious to anyone who has tried to actually solve the Bethe equations. As one increases the spin chain length, one can notice that the Bethe roots start to condense into curves, some examples are shown in fig. 9. This reminds us of the cuts we use to describe solutions in string theory, therefore the hope is that the spectral curve emerges out of the Bethe ansatz equations in the asymptotic limit $L \to \infty$ keeping $J \sim L$. Taking the logarithm of the Bethe equations in (3.50) produces

$$L \log \left(\frac{u_{i,k} + \frac{i}{2} V_k}{u_{i,k} - \frac{i}{2} V_k} \right)^L = \sum_{l=1}^{r} \sum_{\substack{j=1 \\ j \neq i}}^{J_l} \log \left(\frac{u_{i,k} - u_{j,l} + \frac{i}{2} M_{kl}}{u_{i,k} - u_{j,l} - \frac{i}{2} M_{kl}} \right) - 2\pi i n_{ik}, \quad (3.93)$$

where n_{ik} are mode numbers arising due to the multivalued nature of the logarithm. Taking the asymptotic limit and rescaling the Bethe roots by $x_{i,k} = u_{i,k}/L$ yields

$$-\frac{V_k}{x_{i,k}} = \sum_{l=1}^{r} \sum_{\substack{j=1 \\ j \neq i}}^{J_l} \frac{1}{J_l} \frac{M_{kl}}{x_{i,k} - x_{j,l}} - 2\pi n_{ik}. \quad (3.94)$$

Introducing the root density

$$\rho_k(x) = \frac{1}{J_k} \sum_{j=1}^{J_k} \delta\left(x - x_{j,k}\right) \quad (3.95)$$

brings the Bethe equations to their continuum form of

$$-\frac{V_k}{x} = \int_{\mathcal{C}} dx \, \frac{\rho_k(x) M_{kf(x)}}{x - y} + 2\pi n_{ik}, \quad (3.96)$$

where we now integrate along all of the cuts and $f(x)$ is an auxiliary function taking on the value of k, the number of the cut that we integrate along. One can now introduce the resolvet

$$G_k(x) = \sum_{j=1}^{J_k} \frac{1}{x - x_{j,k}} + \frac{V_k}{x} \simeq \int_{\mathcal{C}_k} dy \, \frac{\rho_k(y)}{y - x} + \frac{V_k}{x} \quad (3.97)$$

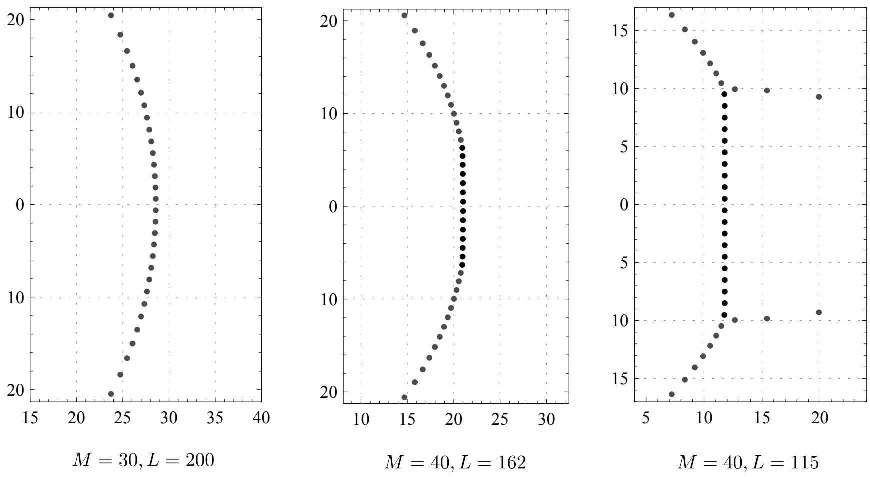


Figure 9: Examples of Bethe roots condensing to cuts on the complex plane. The roots correspond to M excitation states in spin chains of length L. The mode number is $n = 1$. [62]

and rewrite the last equation as

$$M_{kk}\left(G_k(x+i\epsilon) - G_k(x-i\epsilon)\right) + \sum_{j\neq k} M_{kj} G_j(x) = 2\pi n_{jk}. \tag{3.98}$$

Finally introducing quasi-momenta by $p_i \sim \pm\left(G_{i-1} - G_i\right)$ we arrive at the condition [63]

$$p_i(x+i\epsilon) - p_j(x-i\epsilon) = 2\pi n_{ij}, \tag{3.99}$$

which indeed resembles the condition (3.81) we found when discussing the string spectral curve. A detailed comparison has been made between the string spectral curve and the gauge spectral curve [64] and they seem to be in good agreement once again confirming the AdS/CFT correspondence.

3.3 Further developments

Up to now we have been discussing integrability in the context of the spectral problem in AdS/CFT and eventually we described how integrability solves it completely. The methods we discussed and their applicability domains are shown in fig. 10. The full solution is the Y-system with all other approaches being limiting cases. We gave a lot of references for tests of the solution and as already mentioned before, all of them indicate that it is correct. However

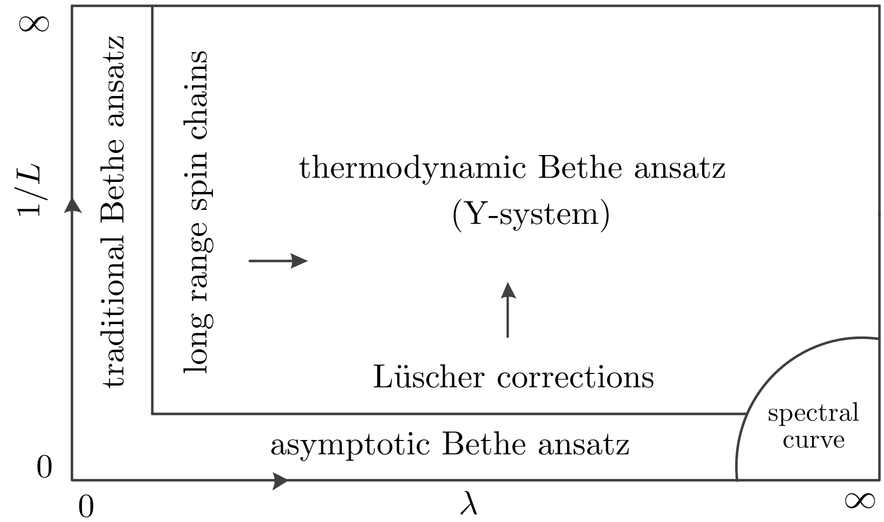


Figure 10: The various methods of solving the spectral problem in AdS/CFT and their application domains.

there are still areas left to be explored, e.g. we discussed how integrability emerges from string theory in the form of the spectral curve, however it is limited to classical and quasi-classical analysis only. Since the full solution for the spectral problem emerged from the gauge theory side it would be nice to do the same for strings. An even further reaching goal is to fully describe the quantum theory of strings on $AdS_5 \times S^5$, which would be a truly groundbreaking discovery.

Solving the spectral problem is only the first part of solving the conformal field theory completely, since by figuring out the conformal dimensions of all fields in the theory we effectively specify all two-point correlators. In order to reconstruct any n-point correlator in a CFT using the operator product expansion we also need all three-point correlators [26]. In any CFT they are restricted to [65]

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\rangle \sim \frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}\,|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}\,|x_{ki}|^{\Delta_k+\Delta_i-\Delta_j}}, \tag{3.100}$$

where C_{ijk} are some structure constants. If one could find these constants the theory would effectively be solved. Recently some progress has been achieved in this direction using integrability techniques [65, 66].

Another area that has achieved a tremendous amount of attention in AdS/CFT is the subject of scattering amplitudes. The AdS/CFT correspondence relates a scattering problem in gauge theory to a problem of finding a minimal surface in AdS space of a shape that has a polygonal boundary made up of the incoming momenta [67]. The integrability program is then used to reduce this problem even further to a set of integral equations, which can eventually be solved numerically [68]. A lot of work is also being done in trying to understand possible corrections to the current results by going beyond the planar limit [69], even though integrability is thought to break down in this regime. Another direction is trying to find integrable structures in other theories. It has indeed been done in the $\mathcal{N} = 6$ Chern-Simons theory, which is dual to strings on $AdS_4 \times CP^3$ [70]. Theories with less supersymmetry and other types of deformations to $\mathcal{N} = 4$ SYM are also being looked upon [71, 72] with the ultimate goal of gaining more insight into solving QCD. This has also open up a subject on its own in mathematics, which seeks to answer the question of what mathematical structure makes a theory integrable. This has lead to many developments in the theory of quantum algebras and specifically Yangians [73].

4 Conclusions

In this thesis we reviewed the AdS/CFT correspondence and the integrable structures that appear in the dual gauge and string theories, which make the duality even more intriguing. The AdS/CFT duality on its own is a ground breaking discovery that has opened up pathways for many more discoveries in theoretical physics. The very basic idea behind the duality is rather simple – open strings attached to branes can also be viewed as closed strings moving on a curved background. Yet this picture relates two very different theories that don't even share the same number of dimensions and most importantly, theories that have perturbative expansions in completely different regimes. This opens up new possibilities for understanding gauge theories and string theories in general with the hope of getting more insight into the physical theory of strong interactions, QCD. And even though a string dual for QCD is not known, the original AdS/CFT duality provides a very nice playground where various new ideas can be explored.

One of the most unexpected ideas to come out of this playground is integrability, which is discussed in the second part of this thesis. The idea that a highly complicated theory like string theory on $AdS_5 \times S^5$ can be solved exactly is mind blowing. We sketched how integrability aided in solving the spectral problem in AdS/CFT exactly, which is a big step towards solving both of the theories completely. Integrability also provides a very good check on the correspondence, since everything one does in one theory must somehow translate to the other theory – they are after all the same theory. We saw that this indeed happens: spin chains in gauge theory correspond to solutions in string theory, spectra of spin chains match the spectra of strings, etc. It is hard to imagine how this could be a consequence. Most importantly integrability provides an exact solution to the spectral problem in both theories, meaning that results can actually be compared numerically – something that is not possible in perturbation theory. All checks to this date have passed the test, once again confirming the correspondence.

At the end of the thesis we presented a wealth of areas and problems that are still actively being researched upon, these include finding three-point correlators, calculating scattering amplitudes, generalizing integrability results to other theories and many more. So as one can see, even though integrability is a very popular area in modern physics with many great minds working on it, there is still a lot left to discover.

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